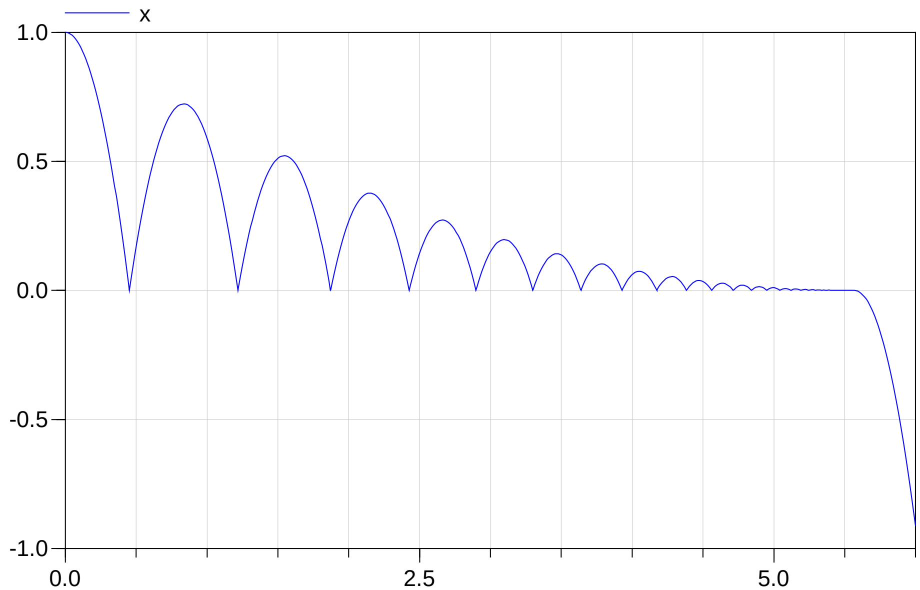
Virtual Physics
Equation-Based Modeling

TUM, January 13, 2015

Modeling and Simulation of Discontinuous Systems



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Motivation

Today, we shall look at the problem of dealing with discontinuities in modeling and simulation.

- Models from engineering often exhibit discontinuities that describe situations such as switching, limiters, dry friction, impulses, or similar phenomena.
- The modeling environment must deal with these problems in special ways, since they influence strongly the numerical behavior of the underlying differential equation solver.

Standard ODE-Solvers

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What happens if we simply apply one of our ODE-solvers on a system with discontinuity?

- The discontinuity occurs in f(**x**(*t*),*t*).
- All ODE-solvers (and their error-estimations) are based on a polynomial approximation of f(**x**(*t*),t).
- Higher-Order methods (order > 1) even suppose that f(**x**,*t*) is differentiable multiple times.

Applying Standard ODE-Solvers

What happens if we simply apply on of our ODE-solvers on a system with discontinuity?

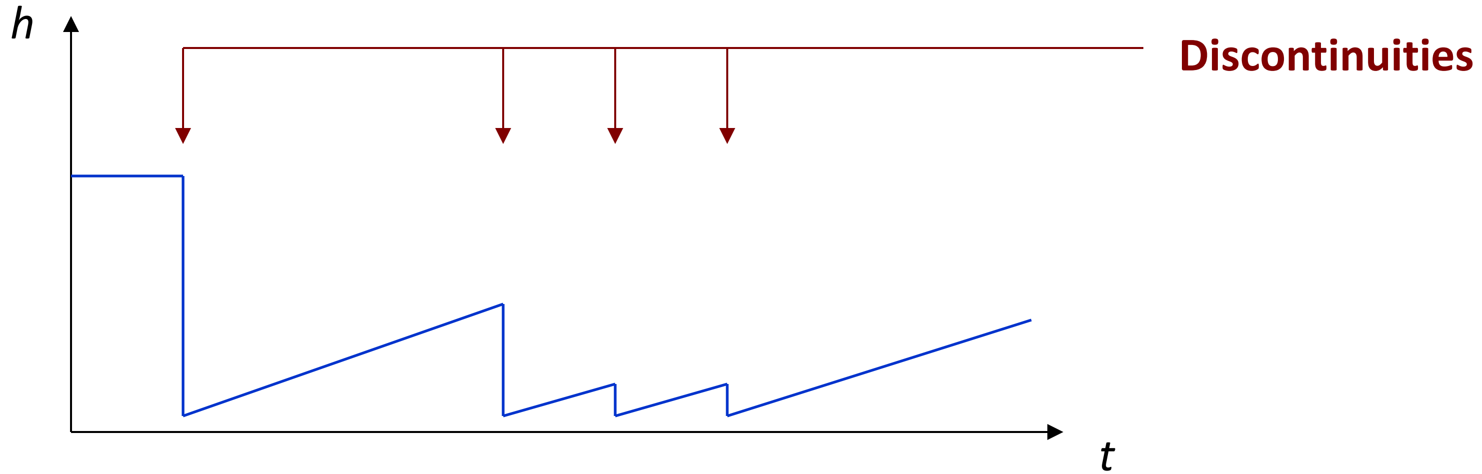
- Polynomials are always continuous and continuously differentiable functions.
- Therefore, when the state equations of the system:

  $$d\mathbf{x}/dt = f(\mathbf{x}(t), t)$$

  exhibit a discontinuity, the polynomial extrapolation is a very poor approximation of reality.
- Consequently, integration algorithms with a fixed step size exhibit a large integration error, whereas integration algorithms with a variable step size must reduce the step size dramatically in the vicinity of the discontinuity.

Applying Standard ODE-Solvers

- An integration algorithm of variable step size reduces the step size at every discontinuity.
- After passing the discontinuity, the step size is only slowly enlarged again, as the integration algorithm cannot distinguish between a discontinuity and a point of large local stiffness (with a large absolute value of the derivative).
- The step-size is constantly too small. The integration is inefficient at best if not even totally inaccurate.



Applying Standard ODE-Solvers

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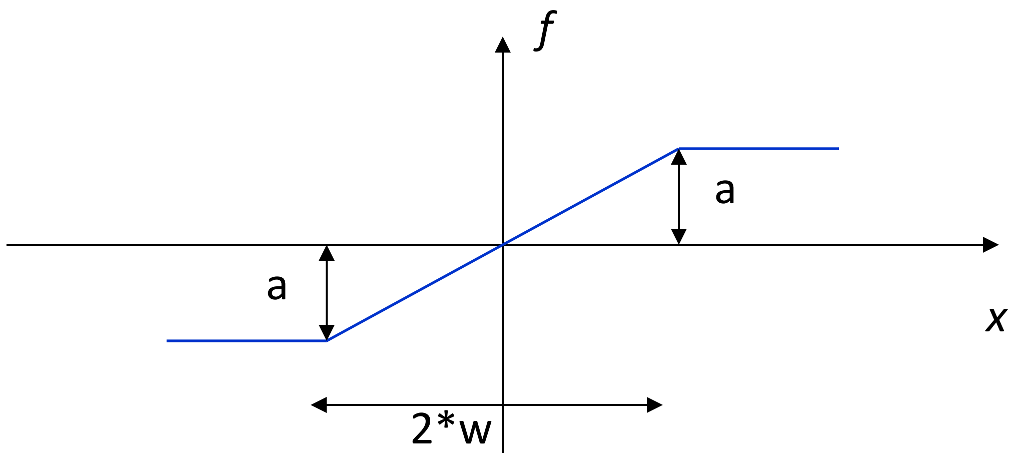
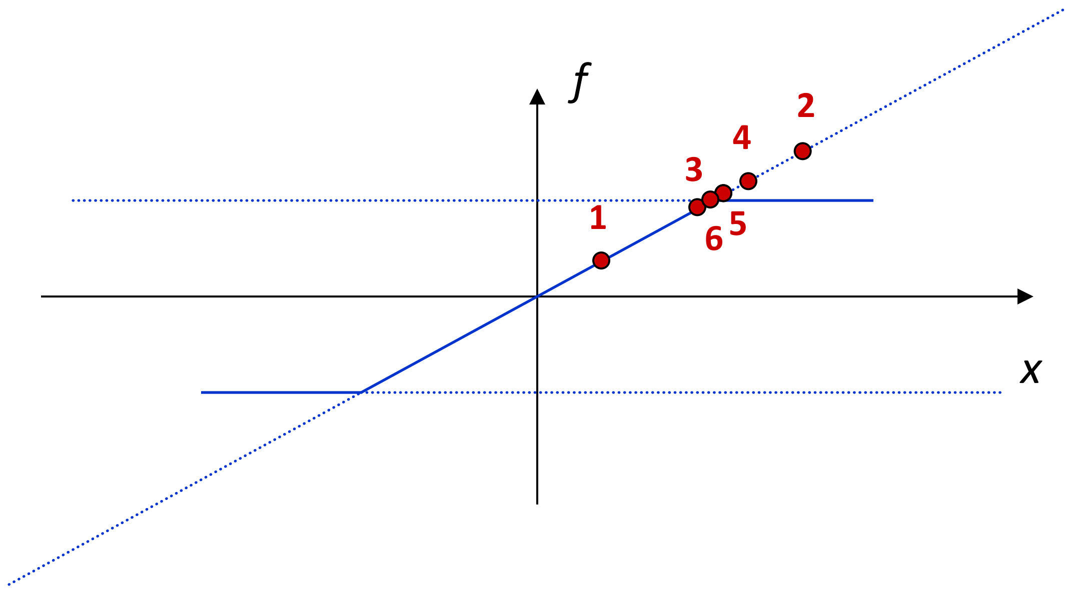
- Trying to handle discontinuities implicitly by standard ODE solvers is evidently not a good solution.
- We can avoid the occurring problems if we model the discontinuities explicitly.
- The expression is one way to do this in Modelica:

```
f = if x < -w then –a else if x<w then a*x/w else a
```



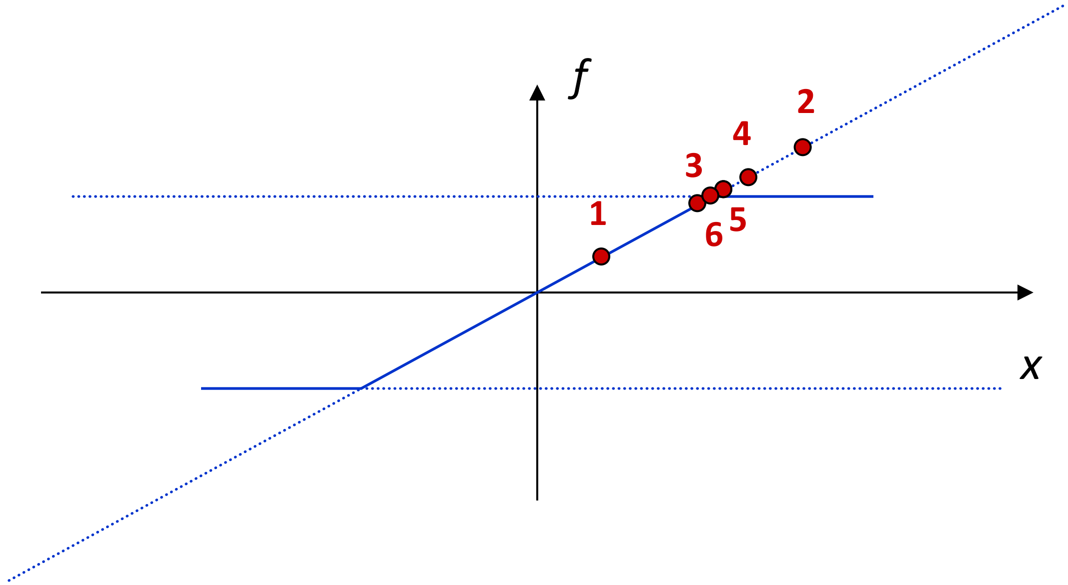
Applying Standard ODE-Solvers

- This if-statement models a state-event since the occurrence of the discontinuity is dependent on the state x.
- An integration algorithm may now precisely locate the event by iterating for the event.
- For instance, by using the bi-section algorithm:



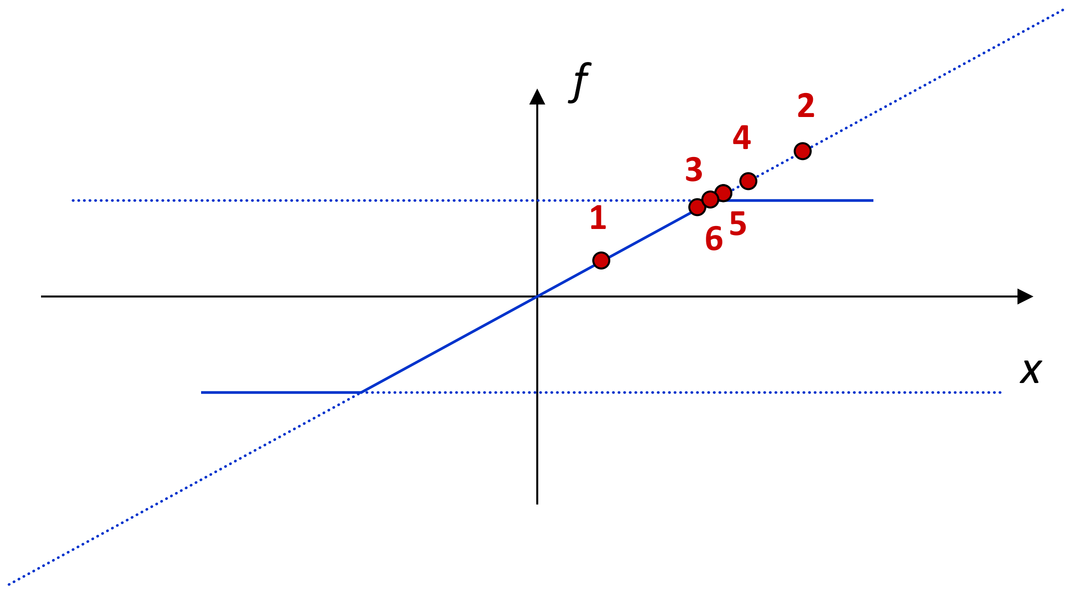
Applying Standard ODE-Solvers

- Bi-section converges slowly. Hence one may prefer the secant method or its “safer”-twin: regula-falsi.
- It is possible to combine the secant method and bi-section
  → Dekker’s methods, Brent’s method.



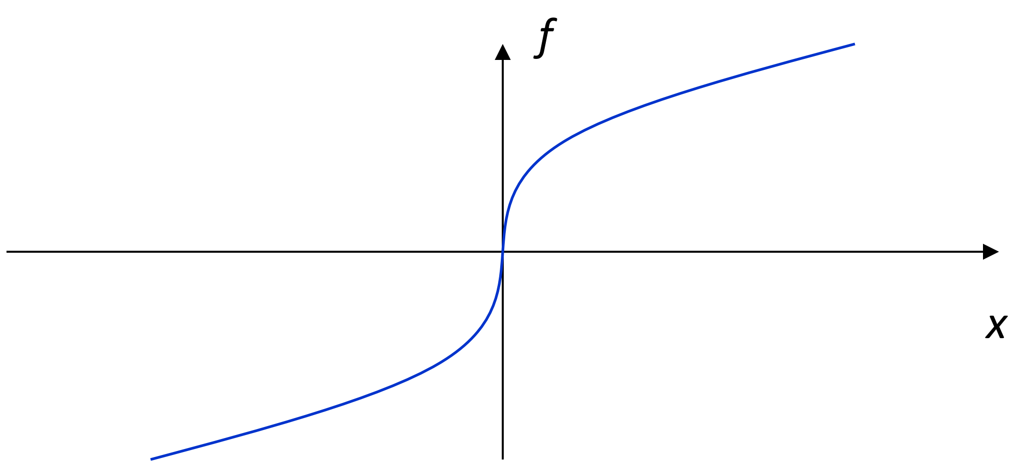
Applying Standard ODE-Solvers

- The iteration for the event-location is thereby performed on the current model equation (here `f = a*x/w`).
- The event itself changes then the model equation (here to `f = a`)



Applying Standard ODE-Solvers

- Sometimes, the event iterations can cause errors in the evalution (division by zero, negative roots)
- Sometimes, the if-statement is used to model continuous functions.
- Hence the noEvent() clause exists: Example:
  ```
  f = noEvent(if x > 0 then sqrt(x) else -sqrt(-x));
  ```
- Here, an event iteration would be both, unnecessary and dangerous. Handling this function is now left to step-size control.



Applying Standard ODE-Solvers

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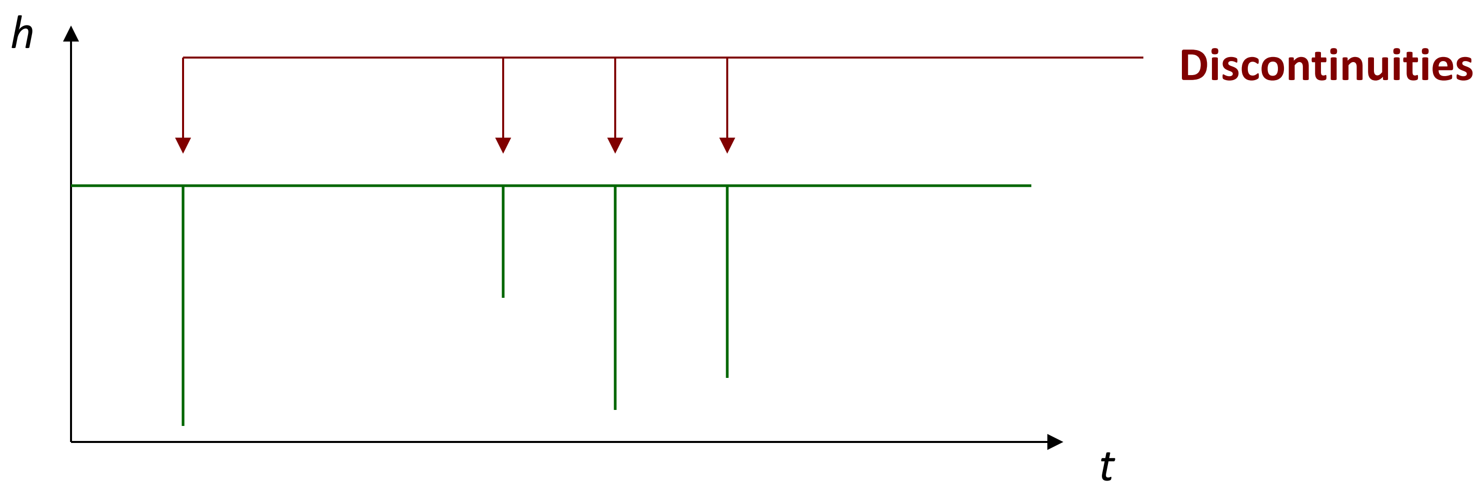
- If we know the precise location of the event, it is sufficient to reduce the size of one single step.
- After passing the discontinuity we switch the model equation and continue with the former step-size.



Applying Standard ODE-Solvers

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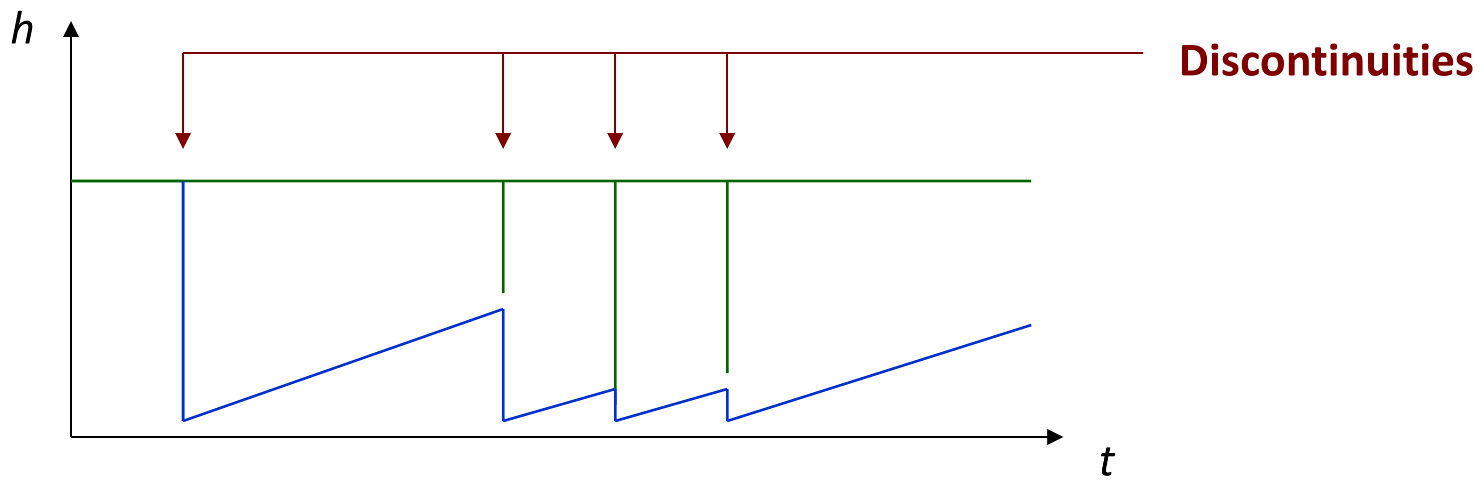
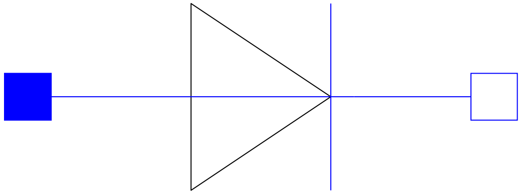
- Evidently, this is much better than abusing step-size control for the treatment of discontinuities.
- We can take much larger step-sizes.



Modeling a Diode

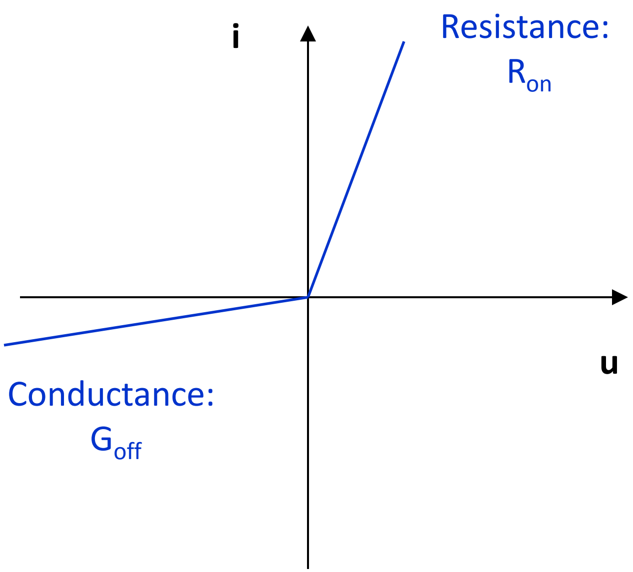
Let us see what we can model with the if-statement.

- For instance, the model of an electrical diode with the following curve.



- Here is one way to model it:

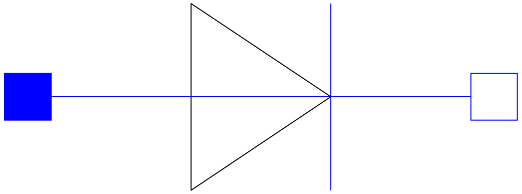
```
u = R*i
R = if u>0 then R_on else 1/G_off;
```



Modeling a Diode

A more compact form is also possible:

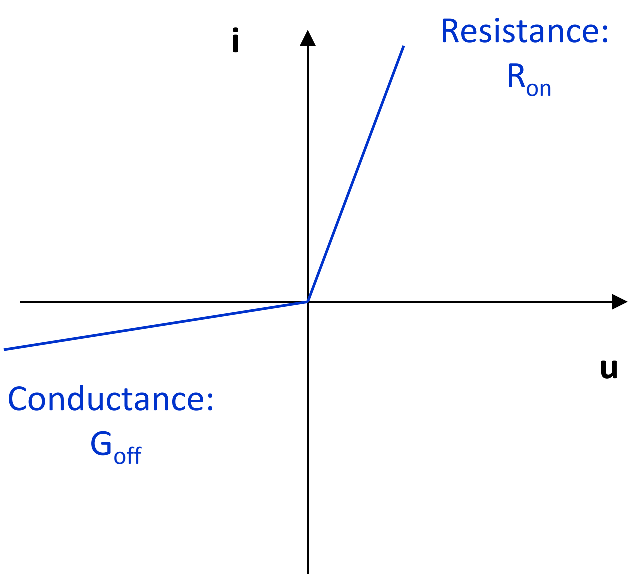
```
u = if u>0 then R_on*i else i/G_off;
```



- This is possible because if-expressions are non-causal in Dymola.
- Internally, the if-expressions may be translated into:

```
u = s*R_on*i + (1-s)*i/G_off;
```

with
s = 1 if u>0
s = 0 if u<0



- The equation can be solved for u or i.

Modeling an Ideal Diode

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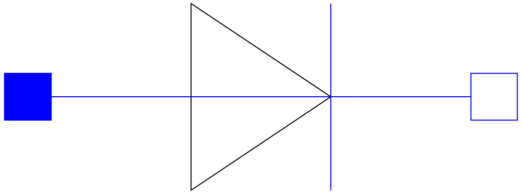
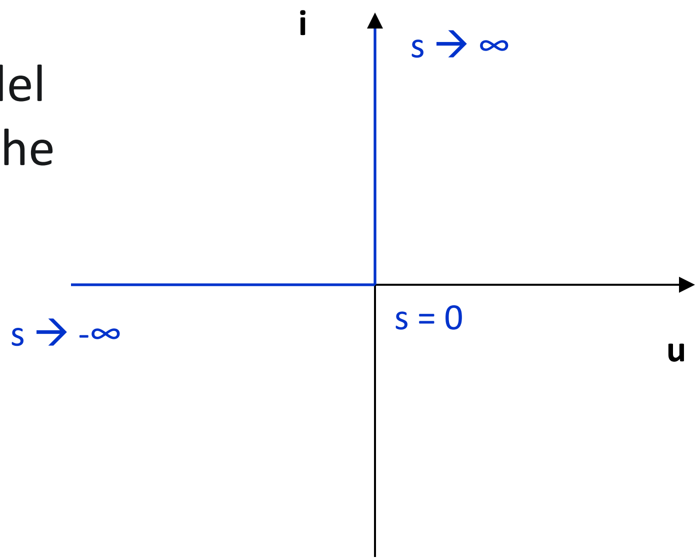
Unfortunately, a truly ideal diode cannot be modeled in this way.



- R_{on} and G_{off} are 0 for an ideal diode.
- The model would be singular in either case.
- We need a different approach. Let us model the diode by a parameterized curve with the curve parameter s.



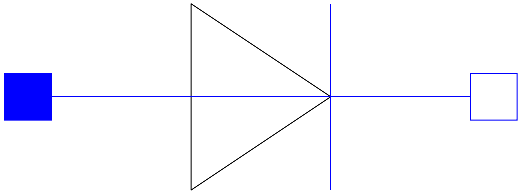
Blocking diode u=s with s < 0

Open diode i=s with s > 0

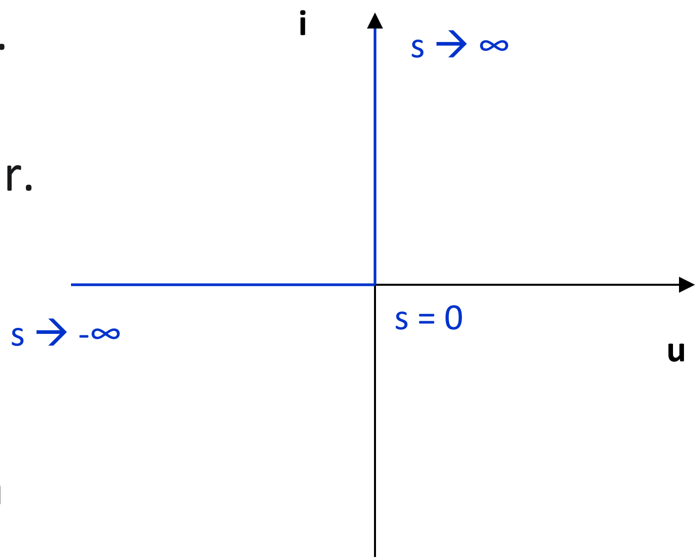
Modeling an Ideal Diode

Here are the corresponding model equations

```
u = if s>0 then 0 else s;
i = if s>0 then s else 0;
```



- These are 2 equations over 3 variables.
  Which are the 2 unknowns?
  If u is known the model is singular for u=0.
  If i is known the model is singular for i=0.
  Only if s is known the model will be regular.

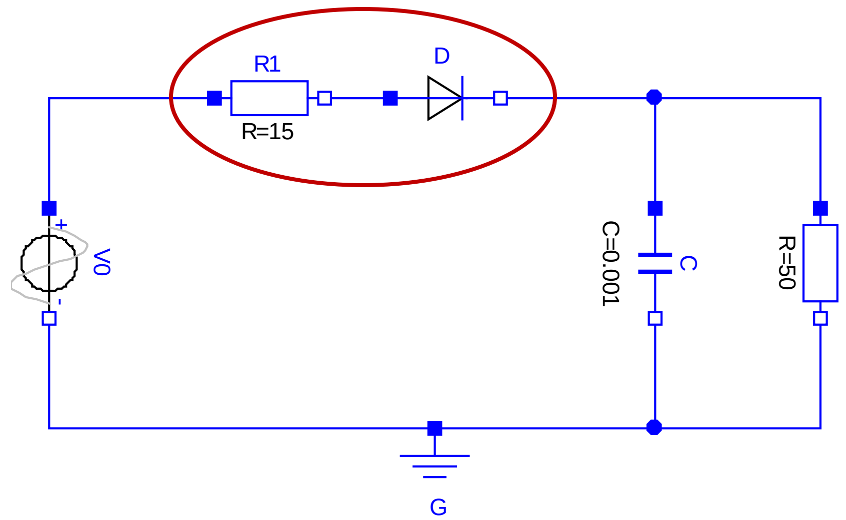


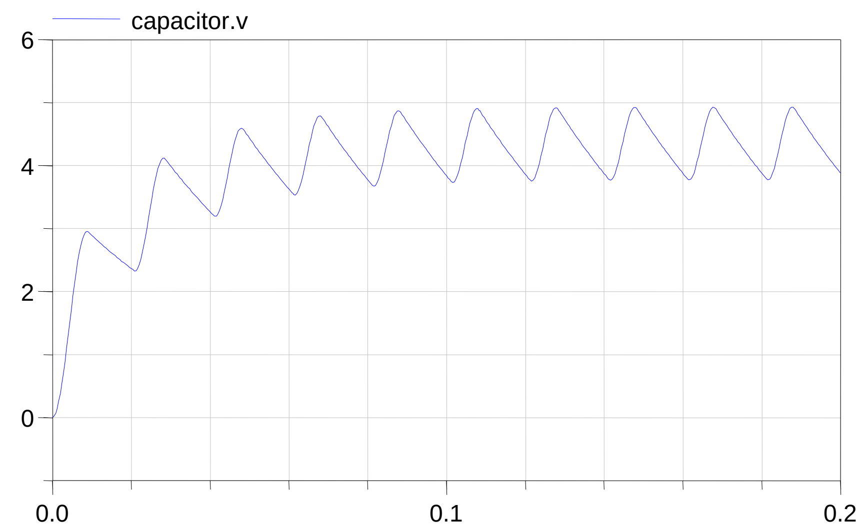
- But s depends itself on u and i. Hence the model needs to be placed in an algebraic loop and s must be chosen as tearing variable of this loop. (Fortunately, Dymola has an in-built heuristics for this…)

Halfway-Rectifier

Here is an appropriate example: the halfway-rectifier.

- The ideal diode D and the resistor R1 form an algebraic loop that determines the voltage drop between source and capacitance.
- The tearing-variable is the curve-parameter s.





Halfway-Rect. w. Line-Inductance

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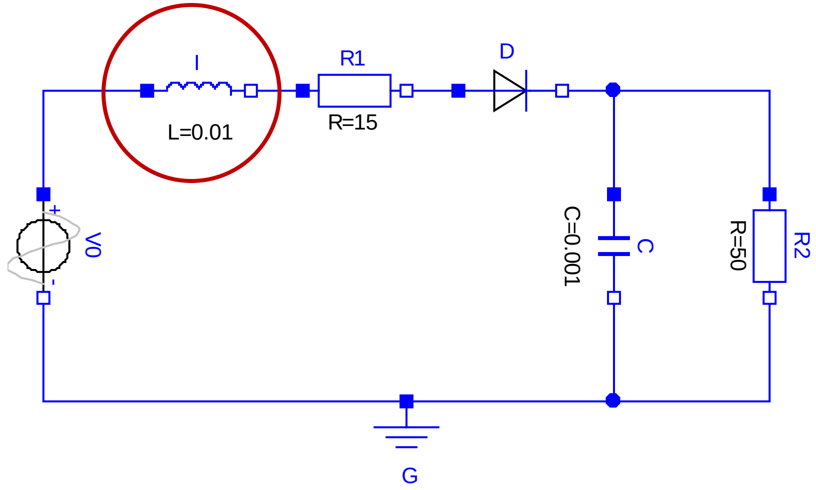
But if we modify this circuit slightly we run into a serious problem.

- We add an inductance in front of the diode.
- Since the natural state-variable of the inductance is the current, the causality of the resistor is fixed and the diode is not part of algebraic loop anymore.
- The simulation fails.
- Let us look closer at this problem.



The inductance L contains the differential equation:

$$di/dt*L = u$$

Hence, i is supposed to be known and the causality of the diode is fixed.

Halfway-Rect. w. Line-Inductance

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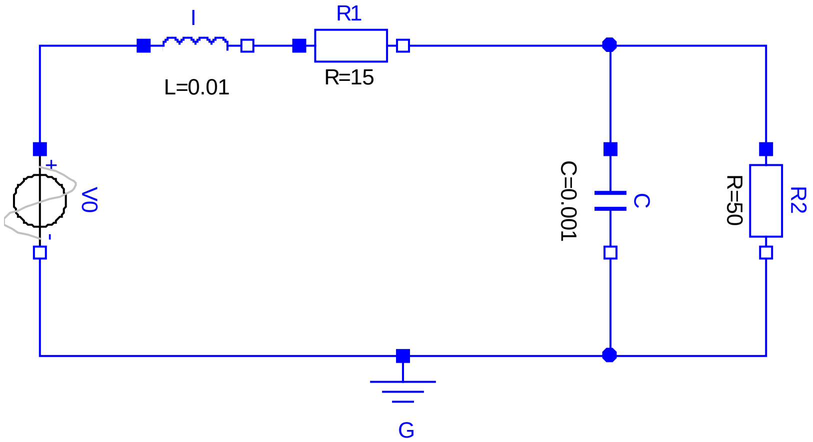
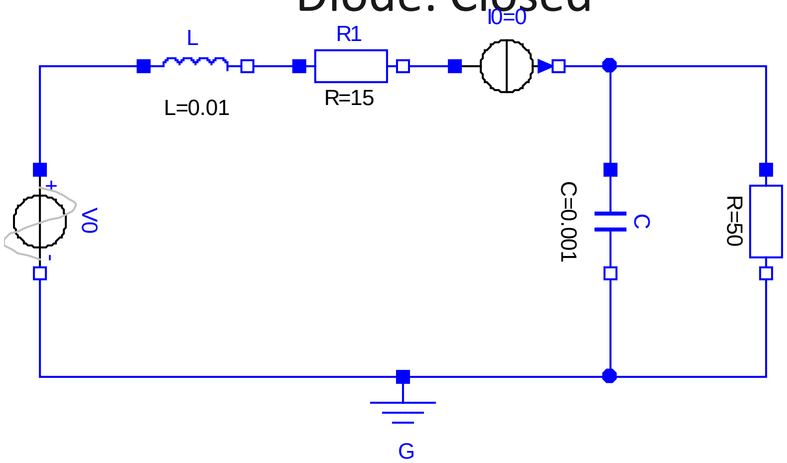
The two circuits below represent the two different states of the diode. (either fully open or fully blocking)

Diode: Open



States (C.v, L.i); Index: 0

Diode: Closed



States (C.v); Index: 1

- The two different states of the diode lead to two different system with different state variables and different perturbation index.
- A severe structural change has been caused by a seemingly harmless equation.
- Dymola is currently unable to handle such variable-structure systems.

Inline-Integration

So what can we do?

- One solution is to use a non-ideal diode and to avoid the structural change at all. However this implements an artificial stiffness into the system that may be unwanted.
- Fortunately, there is another trick: Inline Integration.
- Inline integration means that we inline the time-discrete equation of the integration algorithm into the model equations.
- To this end, we need to replace the corresponding differential equations.

Inline-Integration: Example

Let us use inline integration for the halfway rectifier with line inductance.

- We want to inline Backward Euler (BE or BDF1) into the model of the inductance.
- Hence the differential equation of the inductance:

$$di/dt * L = u$$

- gets replaced by:

$$(i_t - i_{t-h})/h*L = u_t$$

or

$$i_t = i_{t-h} + u_t/L*h$$

with i_t or u_t as potential unknowns

Inline-Integration: Example

What is the advantage of inline-integration?

- By using inline-integration with BE, we have transformed the equation of the inductance into:

  $$i_t = i_{t-h} + u_t/L*h$$

- This equation is structurally equivalent to a resistor equation. It can be solved for i_t as well as for u_t. Hence it can be also part of an algebraic loop.

- For the halfway-rectifier with line-inductance this means that the equations of the inductance L, the resistor R1, and the Diode D form one algebraic loop using the curve parameter s as tearing variable.

- This kind of inline-integration is also not supported by Dymola. Dymola may perform inline-integration but after the differential index-reduction has taken place. Hence this trick does currently not work in Dymola.

Multi-Valued Functions

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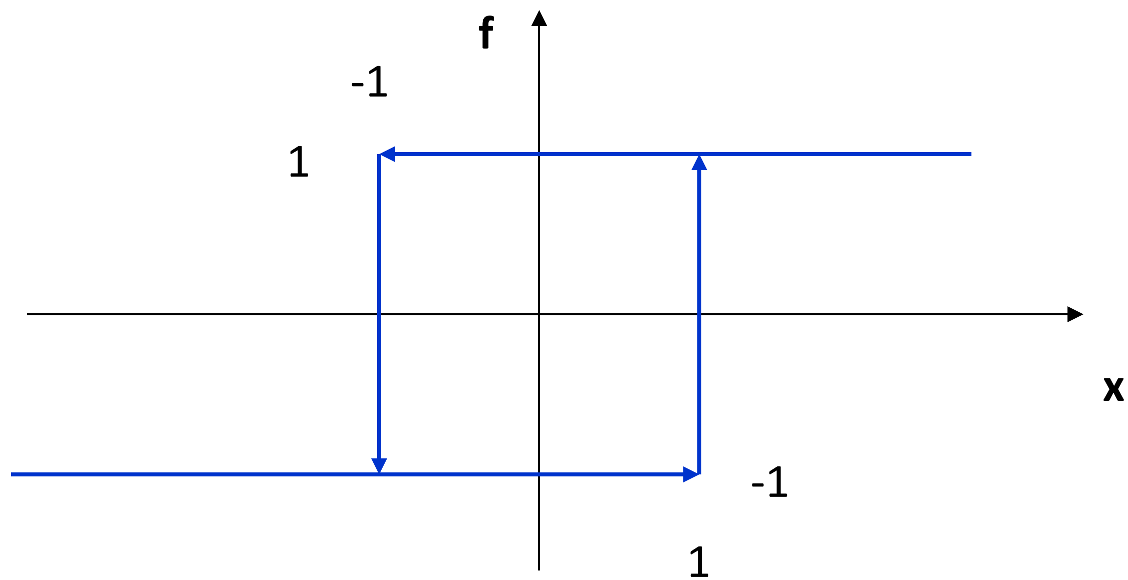
So far, we have only looked at events that could be modeled by if-expressions. However, also multi-valued functions do frequently occur in engineering systems.

- One example is a function for a hysteretic controller (As used, for instance, in a refrigerator or many other devices that require a binary control).



Multi-Valued Functions

To model such functions, the when-statement has been introduced in Modelica.

```
when x > 10 then
        y = -10
end when;
```

- The when statement becomes active exactly when its condition becomes true.
- The equation is rather an assignment: The unknown must be placed on the left.
- The equation is only active for this particular time-instance. Right after, it is deactivated again.
- The value of the unknown is held constant until the next activation of the same when-statement.

Multi-Valued Functions

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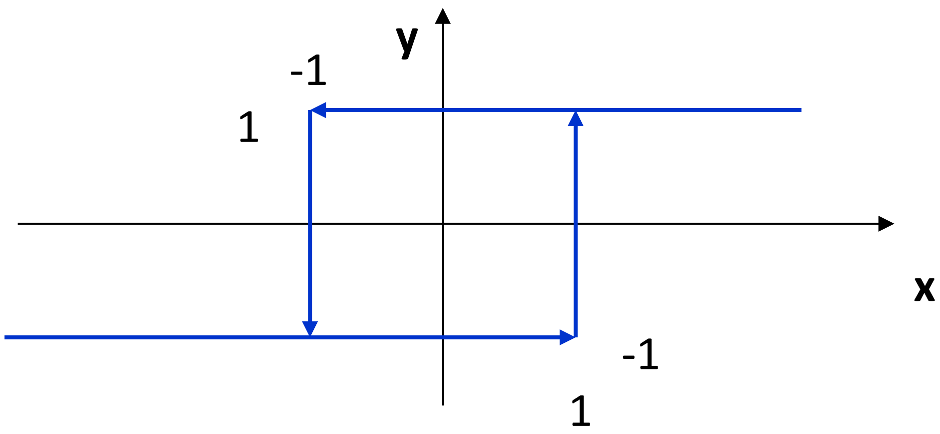
Hence, the following code seems appropriate to model the hysteresis.

```
when x>1 then
      y = 1;
end when;

when x<-1 then
      y = -1;
end when;
```



Multi-Valued Functions

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when x>1 then
      y = 1;
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when x<-1 then
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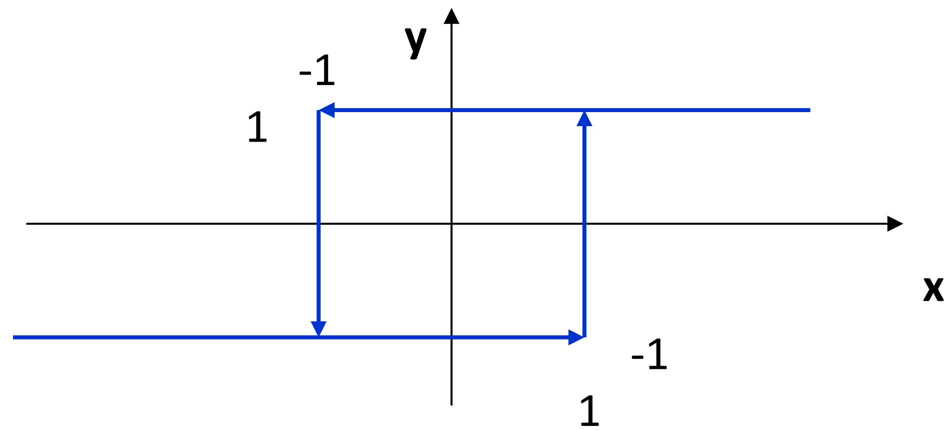
```



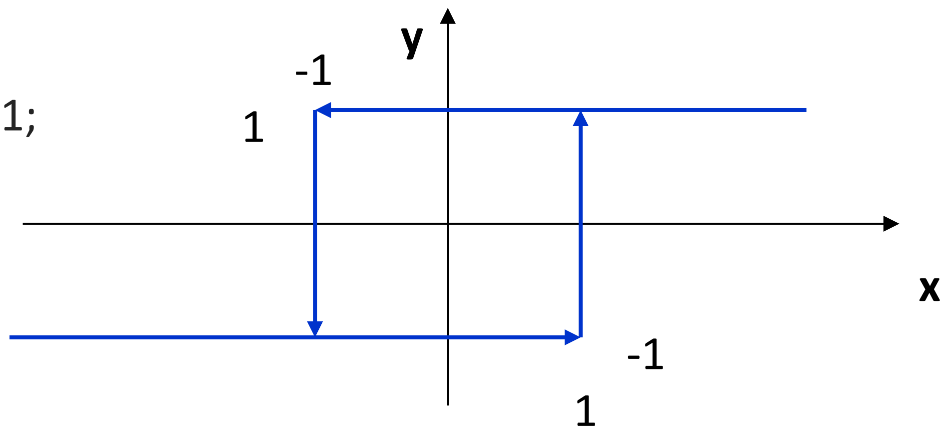
- However, this is illegal in Modelica since the variable y is determined in two distinct when-statements. In order to avoid problems with simultaneous events, this is not allowed.

- Of course, these two events are mutually exclusive, but Dymola does not know this and it is impossible in general to derive this automatically.

Multi-Valued Functions

Here is an alternative formulation:

when x>1 **or** x<-1 **then**
    y = **if** x>0 **then** 1 **else** -1;
end when;

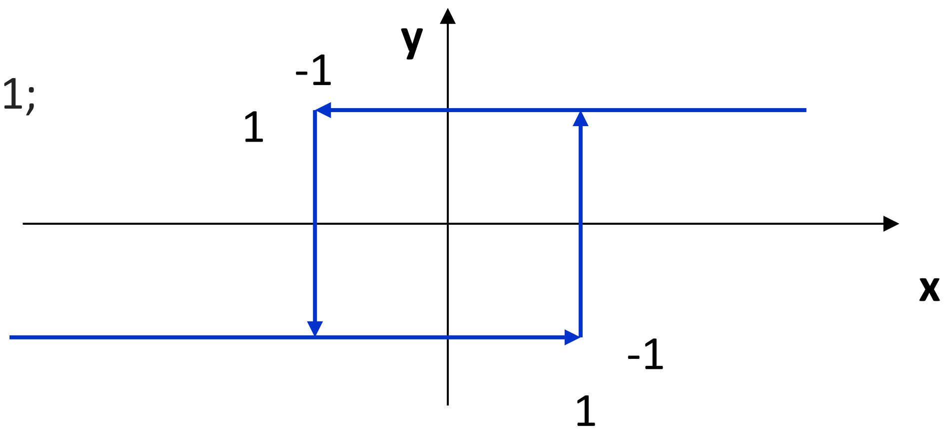


- This is perfectly legal. We have simply merged the two events into a single when-statement.
- By doing so, we have created another problem. Given a large step-size we might jump directly from x=-1 to x=1. In this case, no event is triggered at all.

Multi-Valued Functions

Here is an alternative formulation:

```
when {x>1, x< -1} then
        y = if x>0 then 1 else -1;
end when;
```



- This is perfectly legal. We have simply merged the two events into a single when-statement.
- By doing so, we have created another problem. Given a large step-size we might jump directly from x=-1 to x=1. In this case, no event is triggered at all.
- To cope with this problem, Modelica enables to state a condition-vector. Now, we are fine.

Multi-Valued Functions

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In the Modelica Standard Library, the hysteresis is modeled even differently:

y = **if** x > 1 **or** (pre(y>0) **and** (x>=-1)) **then** 1 **else** -1;

- The operator pre(…) can be used in order to access the value of a variable just right before the event.
- Using this operator, we can formulate multi-valued functions without the use of when-statements.
- In fact, the statement:

  ```
  when g(…) then
     y = f(…);
  end when;
  ```

  is internally transformed to….

  ```
  if g(…) and not pre(g(…)) then
     y = f(…);
  else
     y = pre(y);
  end if;
  ```

Discrete State Changes

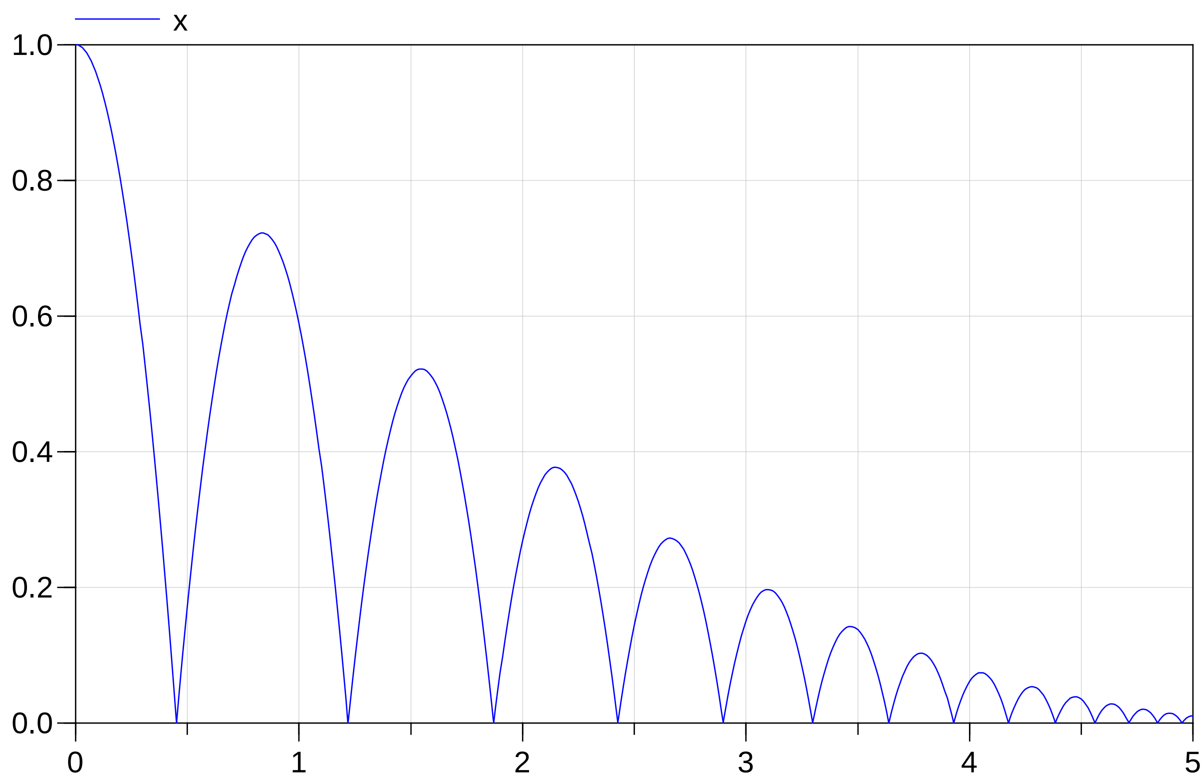
So far, we have only looked at discrete changes in the function $f(\mathbf{x}(t),t)$

$$d\mathbf{x}/dt = f(\mathbf{x}(t),\mathbf{u},t)$$

- But there are also cases where the actual state is changing discretely (e.g. mechanical collisions/impulses) . Here $d\mathbf{x}/dt$ becomes of infinite value. What shall we do?
- This problem corresponds to the re-initialization of the system.
- In current Modelica, this is only weakly supported by the function `reinit(state, newValue)`.
- Let us look at an example: The bouncing ball.

Bouncing Ball

Let us model a bouncing ball that is being dropped from an initial height and is bouncing on a table.



Initializing the Revolute Joint

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Let us model a bouncing ball that is being dropped from an initial height and is bouncing on a table.

- The motion is described by the variables x, v, and a.
- The elasticity of the impulse is determined by the coefficient μ.
- The reinit command is used in a when-clause.
- The pre(…) operator is used to access the prior value of v in order to compute the new velocity.

```
model BouncingBall

  Real x;
  Real v;
  Real a;

  parameter Real mu = 0.85;


initial equation
  v = 0;
  x = 1;

equation
  v = der(x);
  a = der(v);
  a = -9.81;

  when x<0 then
    reinit(v,-mu*pre(v));
  end when;

end BouncingBall;
```

Bouncing Ball

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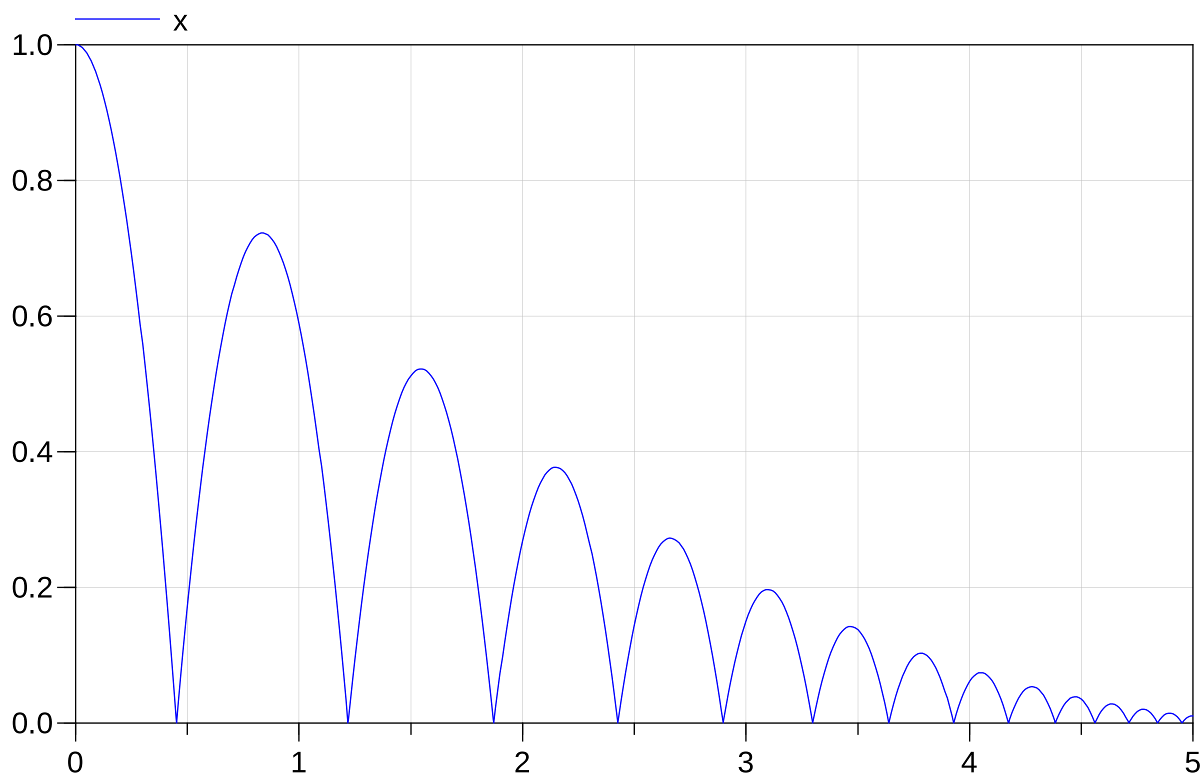
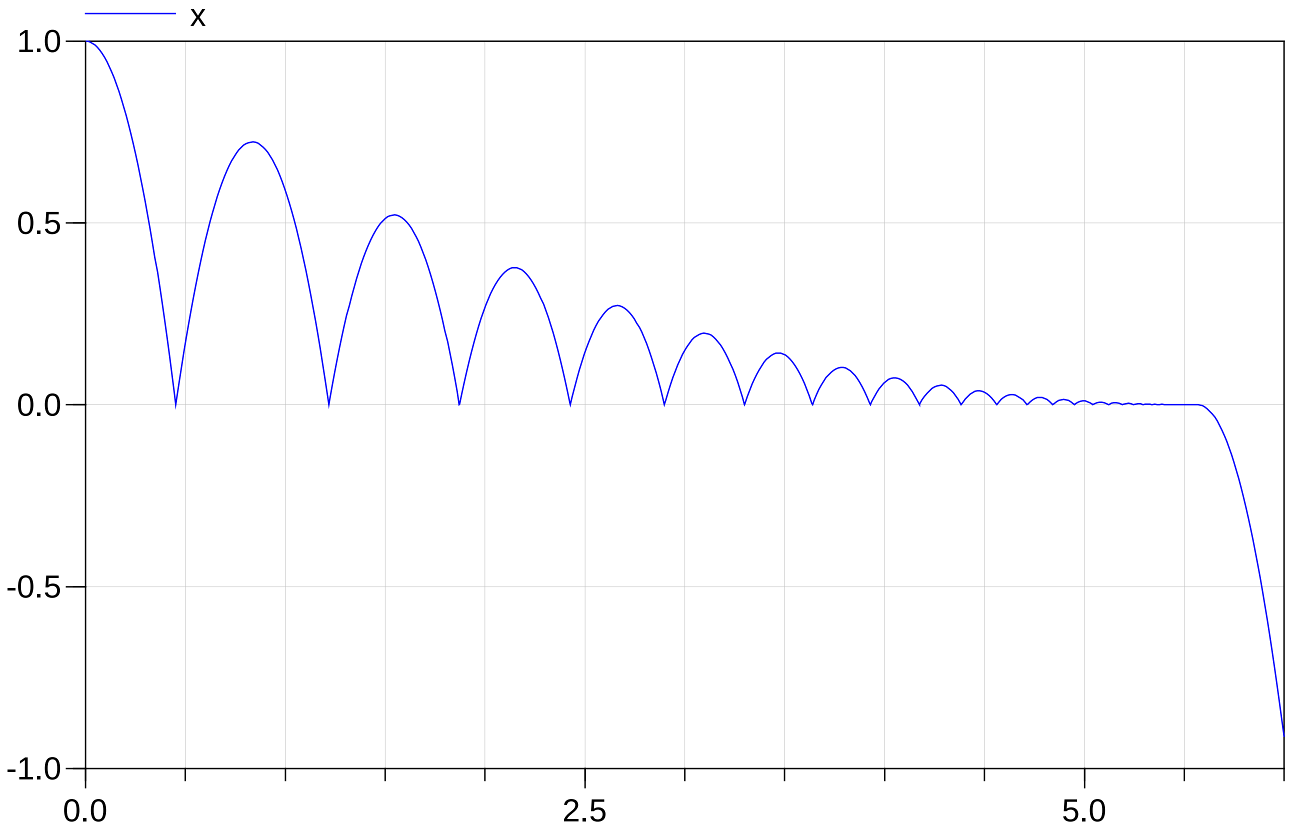
This looks fine. But what happens if we simulate for longer time periods?



Bouncing Ball

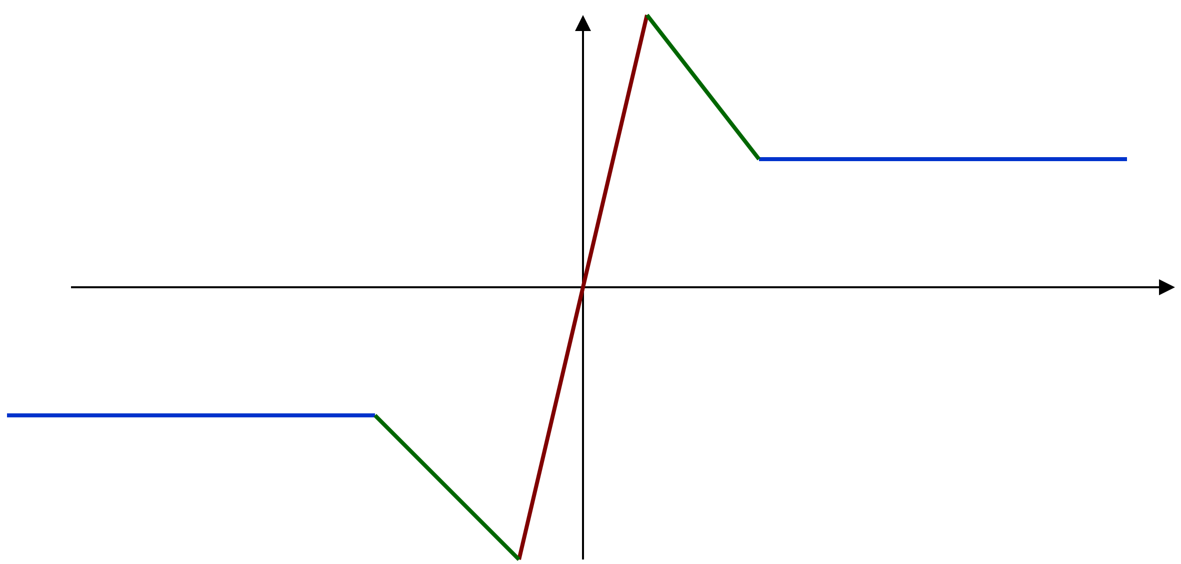
OOOPS!?! This is a common problem among many simulators. The increasingly smaller bounces lead to a failure in the event detection. Modeling a resting state by events is evidently not a good idea.



Dry Friction Revisited

Let us combine what we have learned today by modeling an ideal model for dry-friction.

- For the characteristic curve, we have used so far a regularization. Here is a piecewise linear regularization:



Dry Friction Revisited

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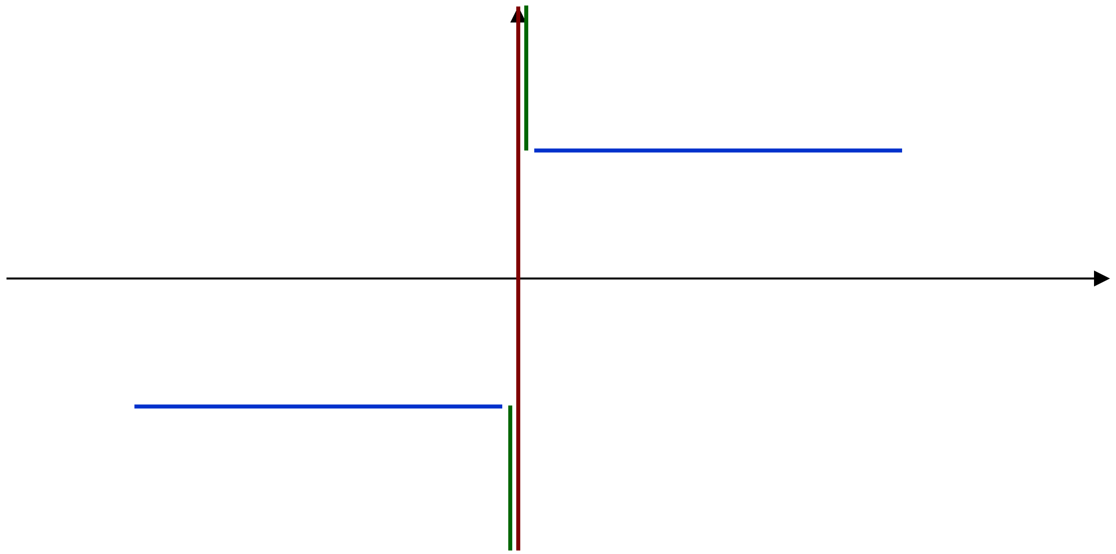
Let us combine what we have learned today by modeling an ideal model for dry-friction.

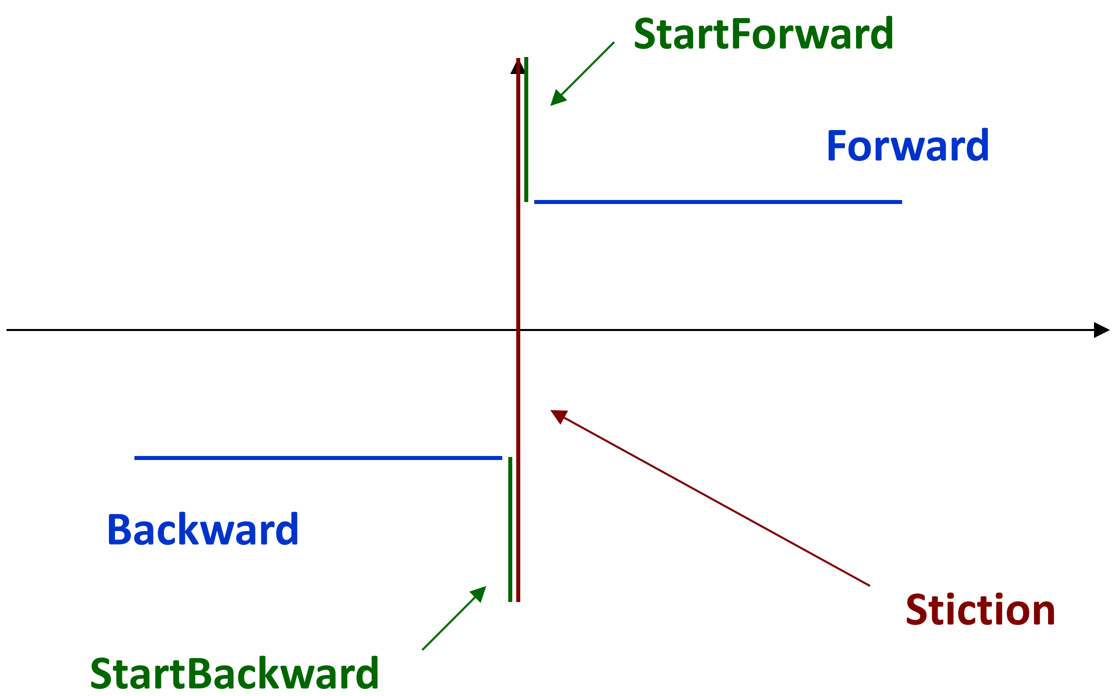
- In the ideal model, this is a multi-valued function.



Modeling Dry Friction

Let us combine what we have learned today by modeling an ideal model for dry-friction.

- In the ideal model, this is a multi-valued function.
- The function contains several modes:



Dry Friction: Mode-Transitions

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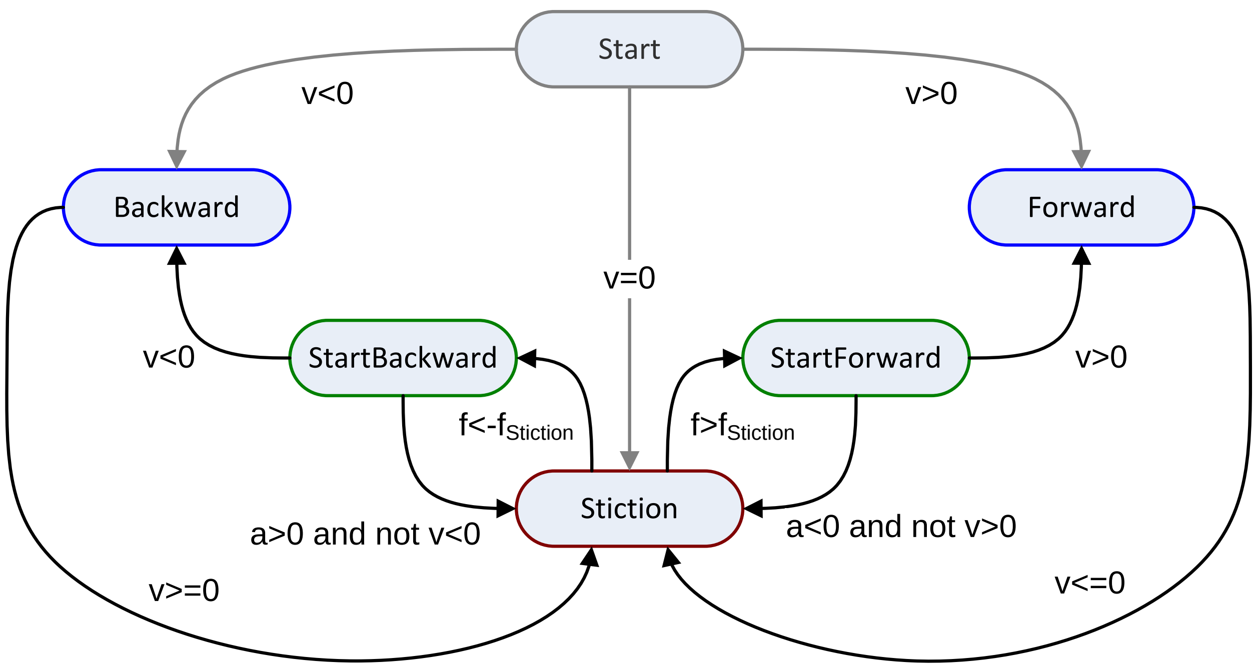
We need to carefully model the transitions between these modes.

- This can be prepared by a mode-transition diagram:



Modeling Dry Friction

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Let us setup the model:

- We use the standard translational interface and derive the velocity and acceleration.
- Two parameters values describe the friction characteristics.
- The modes are represented by a set of Boolean variables.

```
model DryFriction
  parameter SI.Force S = 10;
  parameter SI.Force R = 8;
  Flange_a flange_a;
  SI.Velocity v;
  SI.Acceleration a;
  SI.Force fR;

  Boolean Stiction;
  Boolean StartForw;
  Boolean Forward;
  Boolean StartBack;
  Boolean Backward;


equation
  v = der(flange_a.s);
  a = der(v);
  […]




end DryFriction;
```

Modeling Dry Friction

Let us setup the model:

- The friction force (flange_a.f) is now dependent on the current mode.

```
model DryFriction
  parameter SI.Force S = 10;
  parameter SI.Force R = 8;
  Flange_a flange_a;
  SI.Force fR;
  […]

equation
  […]

  flange_a.f =
    if Forward then R
    else if Backward then - R
    else if StartForw then R
    else if StartBack then -R
    else fR;


  0 =
   if Stiction or initial() then a
   else fR;


end DryFriction;
```

Modeling Dry Friction

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Let us setup the model:

- The friction force (flange_a.f) is now dependent on the current mode.
- The internal operator `initial()` becomes true just at the moment of initialization. Otherwise, it is false.
- Initially or at Stiction, the acceleration is set to zero and the friction force fR is free.
- The conditional constraint a=0 should actually be v=0 at least or s=const, but this would cause a structural change and cannot be handled by Modelica/Dymola.

```
model DryFriction
  parameter SI.Force S = 10;
  parameter SI.Force R = 8;
  Flange_a flange_a;
  SI.Force fR;
  […]

equation
  […]

  flange_a.f =
    if Forward then R
    else if Backward then - R
    else if StartForw then R
    else if StartBack then -R
    else fR;

  0 =
   if Stiction or initial() then a
   else fR;

end DryFriction;
```

Modeling Dry Friction

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Let us setup the model:

- Now we have to model the mode-transitions according to the diagram.
- We can use the pre() operator for this purpose.
- All states must be exclusive.

```
model DryFriction
  parameter SI.Force S = 10;
  parameter SI.Force R = 8;
  Flange_a flange_a;
  SI.Force fR;
  […]

equation
  […]
  Forward = initial() and v > 0 or
            pre(StartForw) and v > 0 or
            pre(Forward) and not v <= 0;
  Backward = initial() and v < 0 or
             pre(StartBack) and v < 0 or
             pre(Backward) and not v >= 0;
  StartForw = pre(Stiction) and fR > S or
              pre(StartForw) and not
              (v>0 or a<=0 and not v>0);
  StartBack = pre(Stiction) and fR<- S or
              pre(StartBack) and not
              (v<0 or a>=0 and not v<0);
  Stiction = not (Forward or Backward or
                  StartForw or StartBack);
end DryFriction;
```

Modeling Dry Friction

Let us setup the model:

- Finally, there is a last issue:
- When the velocity crosses zero and stiction is enforced, we need to set the velocity explicitly to zero.
- To this end, we use the reinit()-command. Hence v must be a state-variable.

```
model DryFriction
  parameter SI.Force S = 10;
  parameter SI.Force R = 8;
  Flange_a flange_a;

  SI.Velocity v(
    stateSelect=StateSelect.always
  );

  […]



equation

  […]

  when Stiction and not initial() then
   reinit(v,0);
  end when;

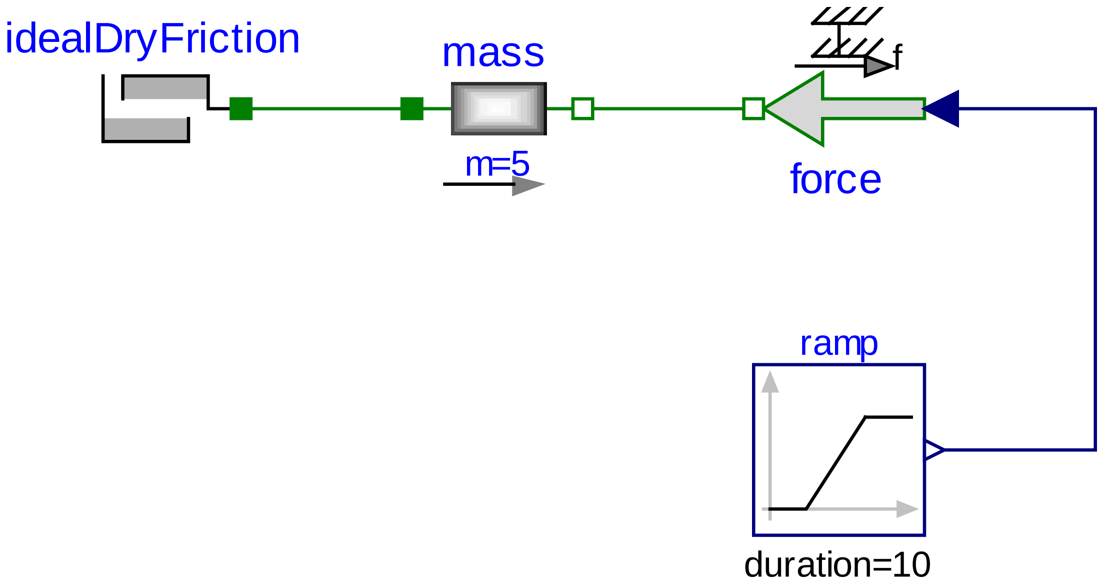


end DryFriction;
```

Simulating Dry Friction

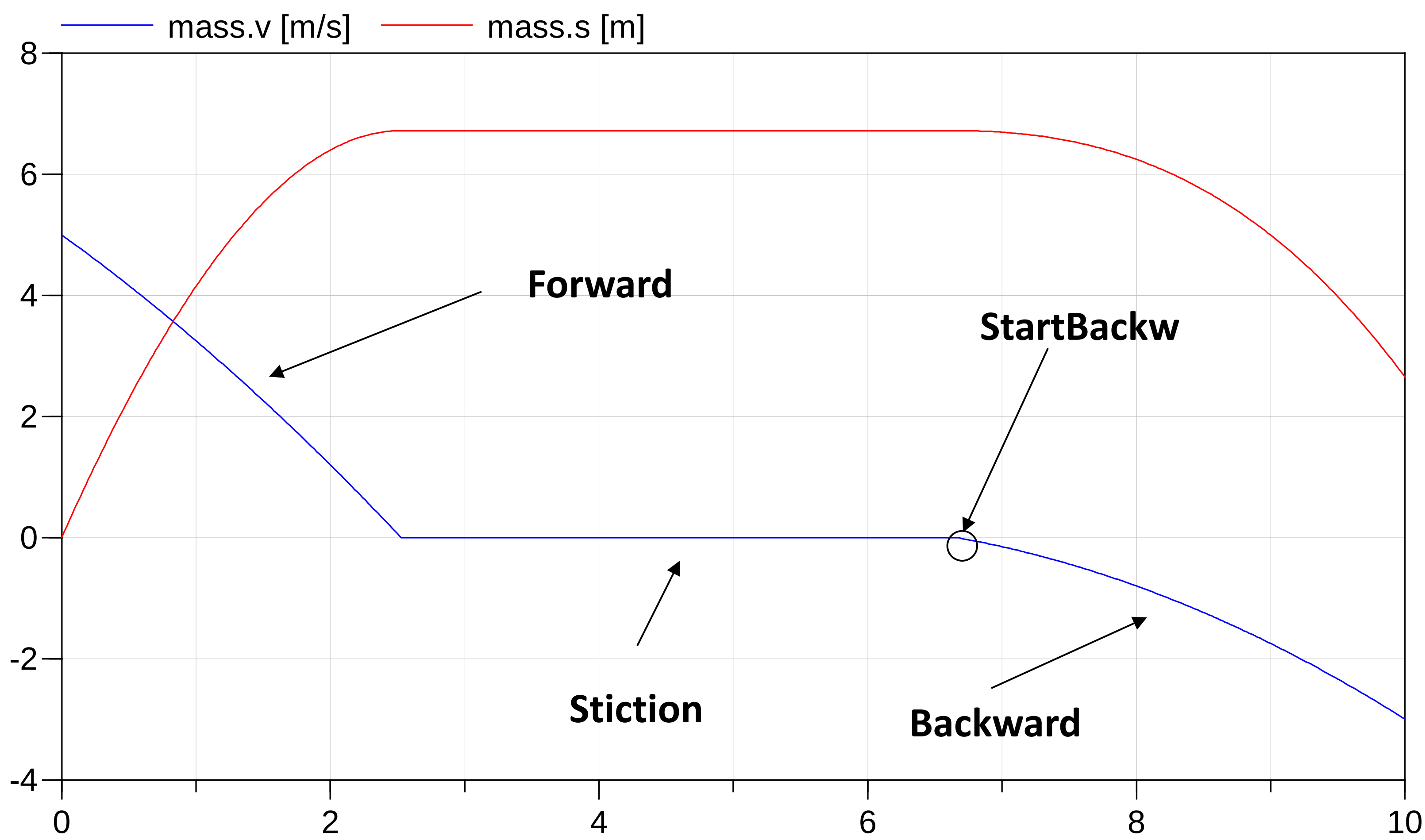
Let us test our dry-friction model:

- The mass (5kg) has an initial speed of 5m/s
- The (negative) force is ramped up from 0 to 15N



Simulating Dry Friction

Here is the simulation result:



Simulating Dry Friction

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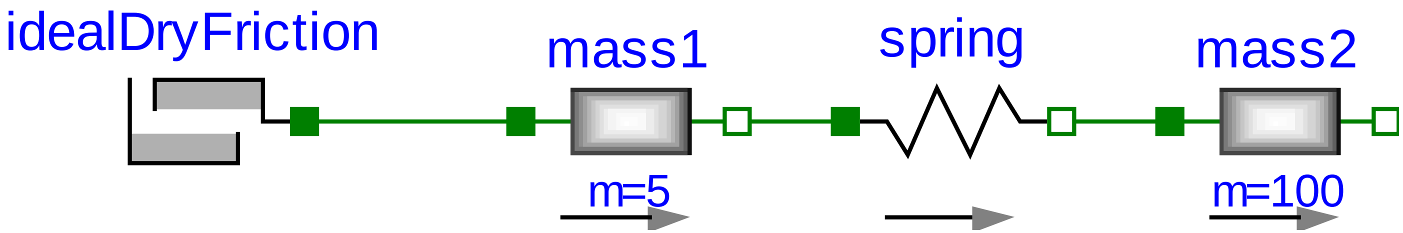
This system is more fun.

- Mass1 (5kg) is initially at rest.
- Mass2 (100kg) starts with v=1m/s.



Simulating Dry Friction

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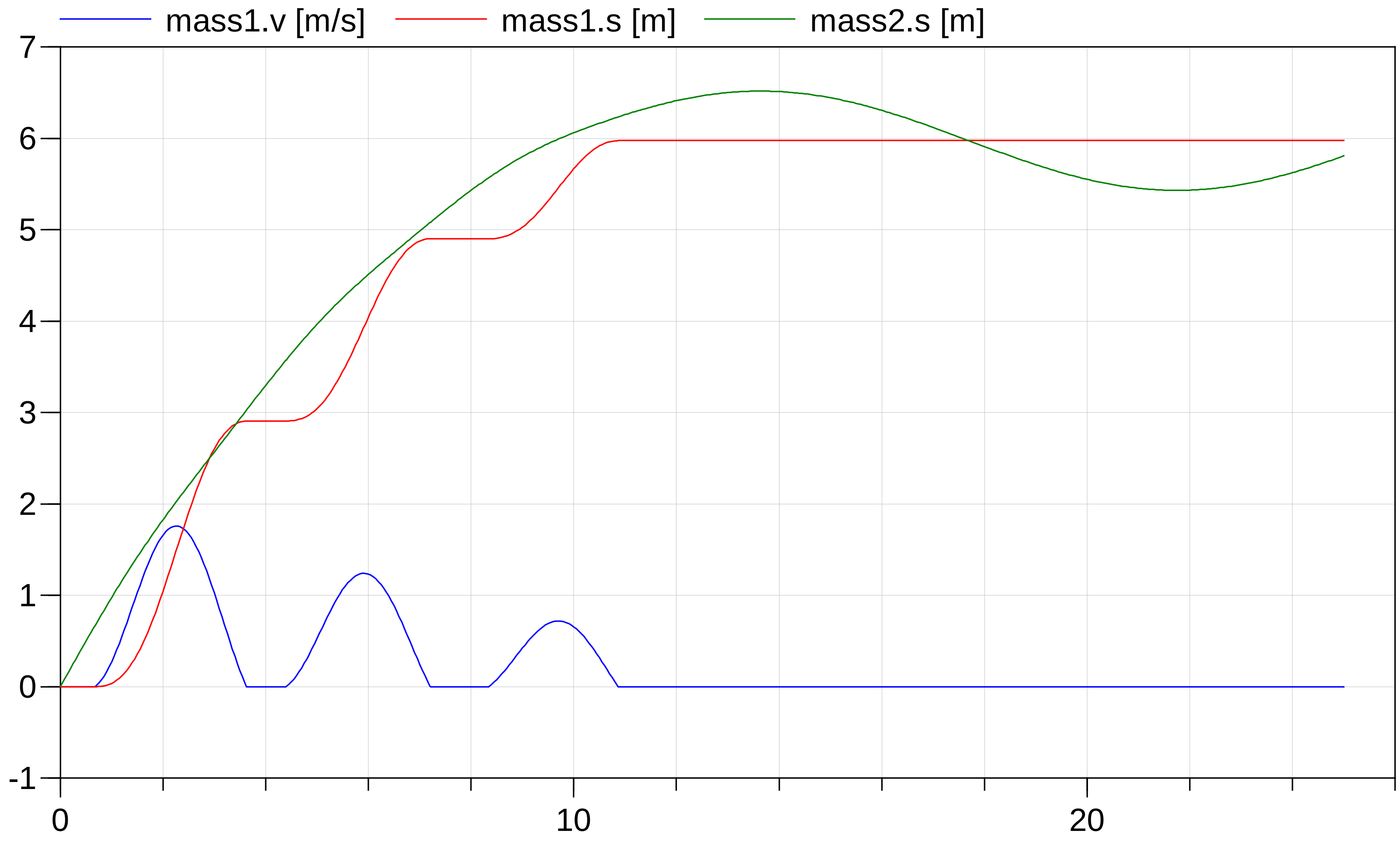
Here is the simulation result:



Questions ?