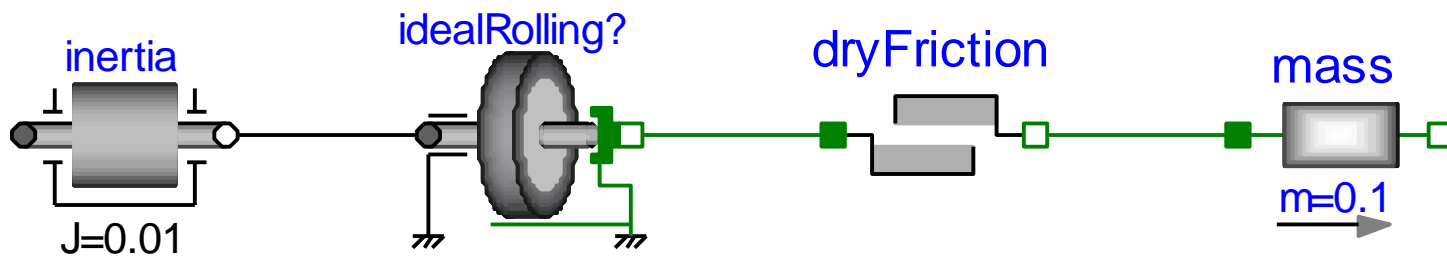


Virtual Physics

Equation-Based Modeling

TUM, November 22, 2022

1D-Mechanical Systems



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For each physical domain, there is a specific pair of effort / flow variables

Domain	Potential	Flow
Translational Mechanics	Velocity: v [m/s]	Force: f [N]
Rotational Mechanics	Angular Velocity: ω [1/s]	Torque: τ [Nm]
Electrics	Voltage Potential v [V]	Current i [A]
Magnetics	Magnetomotive Force: Θ [A]	Time-derivative of Magnetic Flux: $\dot{\Phi}$ [V]
Hydraulics	Pressure p [Pa]	Volume flow rate \dot{V} [m ³ /s]
Thermal	Temperature T [K]	Entropy Flow Rate \dot{S} [J/Ks]
Chemical	Chemical Potential: μ [J/mol]	Molar Flow Rate ν [mol/s]

For the mechanical domain, the first two are relevant:

Domain	Potential	Flow
Translational Mechanics	Velocity: v [m/s]	Force: f [N]
Rotational Mechanics	Angular Velocity: ω [1/s]	Torque: τ [Nm]
Electrics	Voltage Potential v [V]	Current i [A]
Magnetics	Magnetomotive Force: Θ [A]	Time-derivative of Magnetic Flux: $\dot{\Phi}$ [V]
Hydraulics	Pressure p [Pa]	Volume flow rate \dot{V} [m ³ /s]
Thermal	Temperature T [K]	Entropy Flow Rate \dot{S} [J/Ks]
Chemical	Chemical Potential: μ [J/mol]	Molar Flow Rate v [mol/s]

- Each node was represented by a pair of variables

A **potential** variable

v (velocity for translational mechanics)

ω (angular velocity for rotational mechanics)

and a **flow** variable

f (force for translational mechanics)

τ (force for rotational mechanics)

- For one connection between a set of n nodes, n equations have to be generated.

- **$n-1$ equalities**

Translational: $v_1 = v_2 = \dots = v_n$

Rotational: $\omega_1 = \omega_2 = \dots = \omega_n$

- **1 balance equation**

Translational: $f_1 + f_2 + \dots + f_n = 0$

Rotational: $\tau_1 + \tau_2 + \dots + \tau_n = 0$

But the Modelica Standard Library supports different potential variables.

- Not the velocity v but the position s
- Not the angular velocity ω but the angle φ

```
connector Flange_a  
  SI.Position s;  
  flow SI.Force f  
end Flange_a;
```

```
connector Flange_a  
  SI.Angle phi;  
  flow SI.Torque tau;  
end Flange_a;
```

- Why is this? Is our table incorrect?

- No, the table is correct but the correct formulation of mechanical system adds another requirement:

The formulation of holonomic constraints!

- Holonomic Constraints are algebraic constraints on the level of position.
- A rigid rod describes a given distance between two flanges. Here two positions are bound with one constraint equation.
- In order, to formulate such equations correctly, the position needs to be part of the connector.

- Let us model a simple system:
- Two masses connected to springs.
- The position s_1 and s_2 are connected by the following holonomic constraint:

$$s_1 = |s_2| * s_2$$

- Such non-linear constraints are rare in 1D systems but common in multidimensional systems.



- The constraint on positional level:

$$s_1 = |s_2| \cdot s_2$$

- implies for the velocity:

$$v_1 = 2 |s_2| \cdot v_2$$

- Power conservation states

$$v_1 \cdot f_1 + v_2 \cdot f_2 = 0$$

- or

$$f_1 \cdot 2 |s_2| + f_2 = 0$$



thanks to Edo Drenth for the power analysis

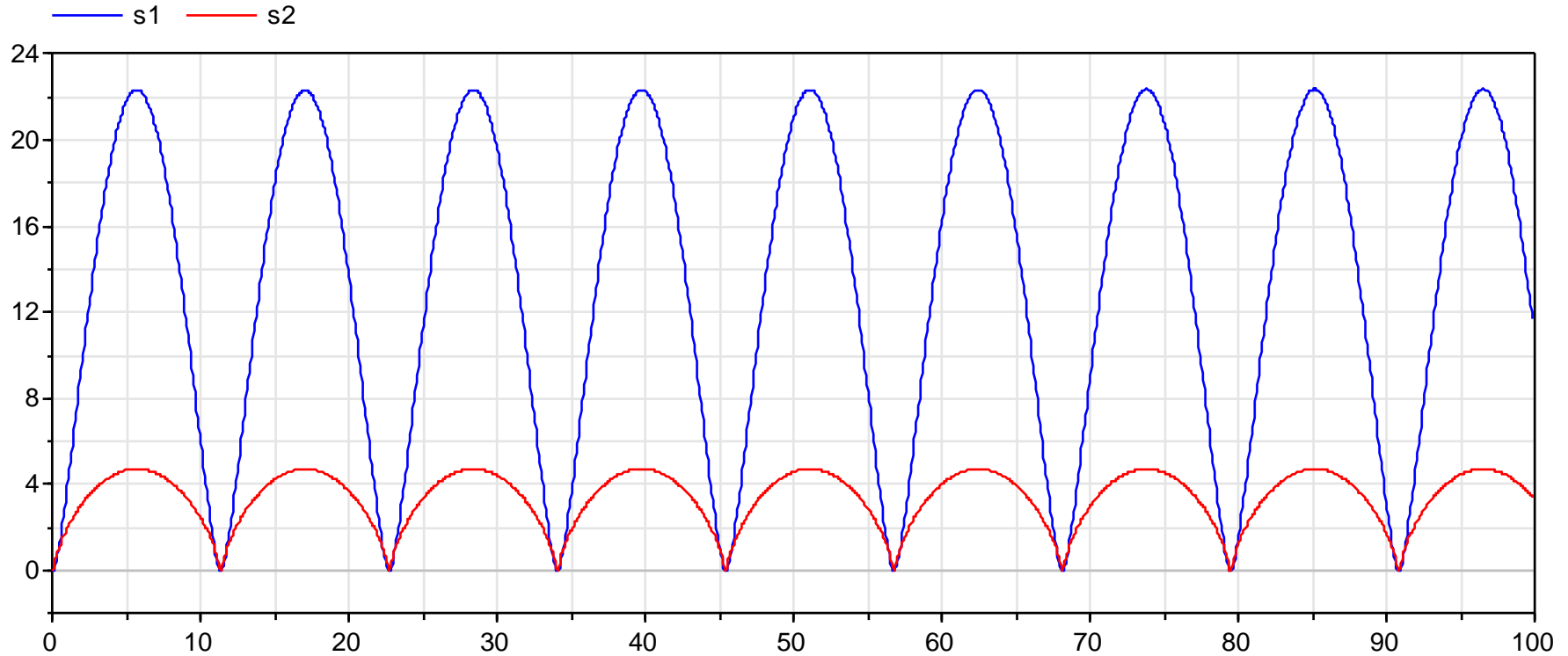
- Here is a handwritten Modelica-code for this example:
- The two variables **s1_int** and **s2_int** are used to formulate the constraints.
- On the next slide you see the simulation result (the positions of the two masses).

```
model TwoSpringsWithConstraint
  Real s1;
  Real s2;
  Real v1;
  Real v2;
  Real f;
  parameter Real m1 = 10;
  parameter Real m2 = 2;
  Real s1_int;
  Real s2_int;
equation
  v1 = der(s1);
  v2 = der(s2);
  -1*s1 + f = m1*der(v1);
  -20*(s2-5) - f*abs(s2_int)*2
    = m2*der(v2);

  s1 = s1_int;
  s2 = s2_int;
  s1_int = abs(s2_int)*s2_int;
end TwoSpringsWithConstraint;
```

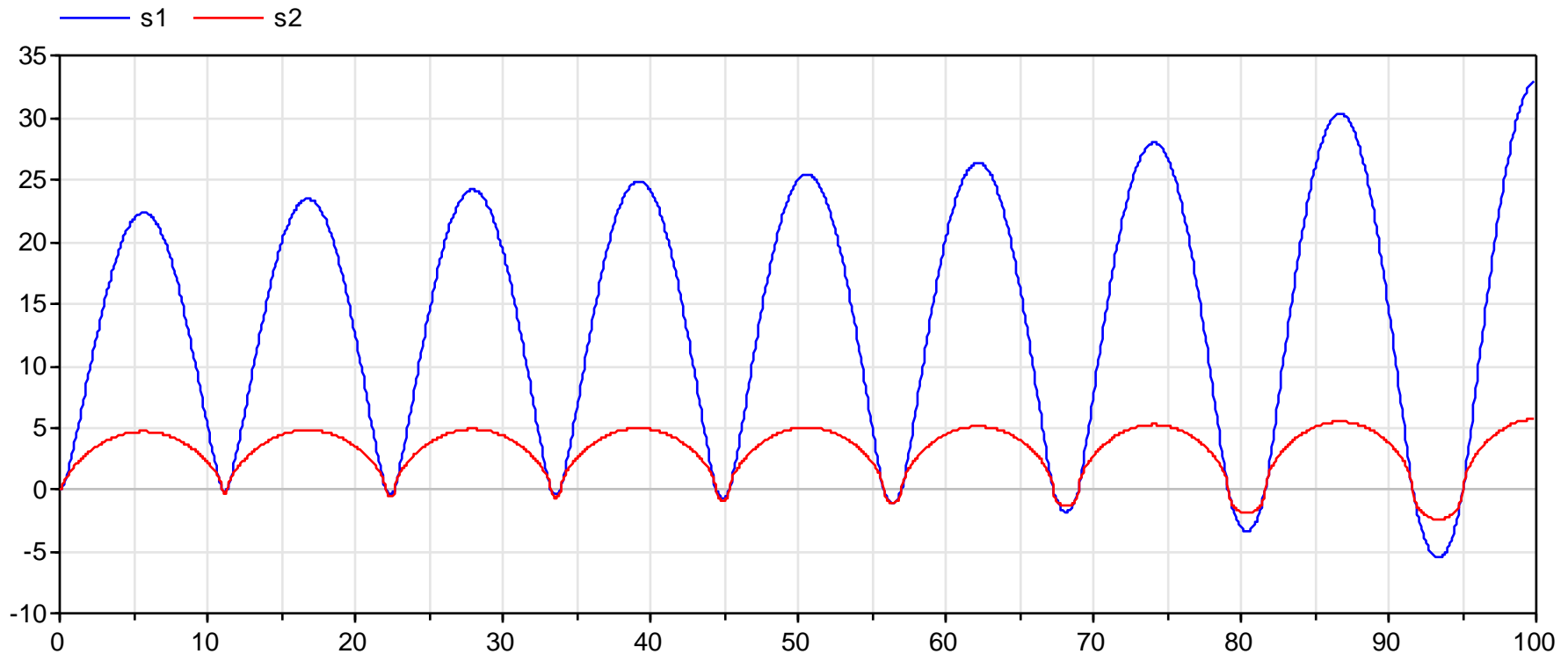
Holonomic Constraints: Example

- With DASSL and a local error tolerance of 0.0001:



Holonomic Constraints: Example

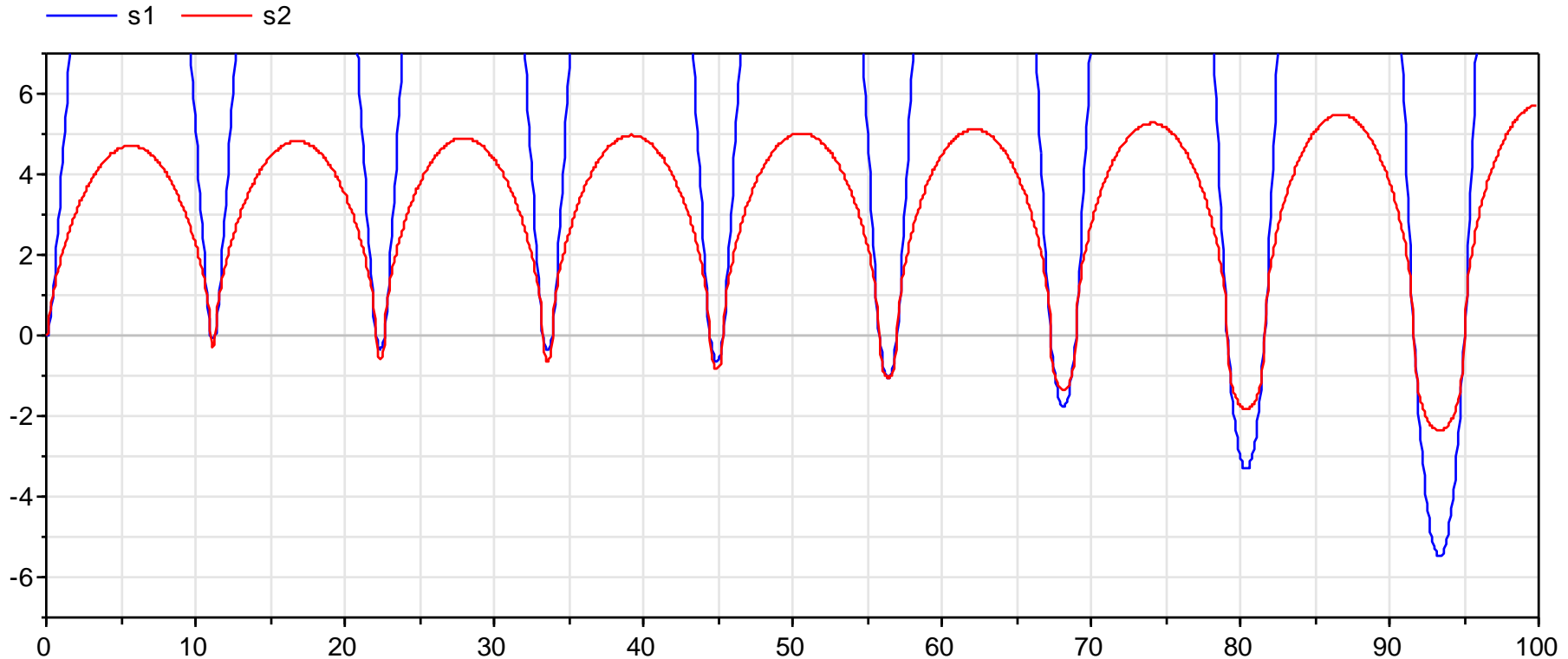
- With DASSL and a local error tolerance of 0.01...



- ... the conservation of energy is violated by numerical integration.

Holonomic Constraints: Example

- With DASSL and a local error tolerance of 0.01...



- However, the holonomic constraint does hold up.

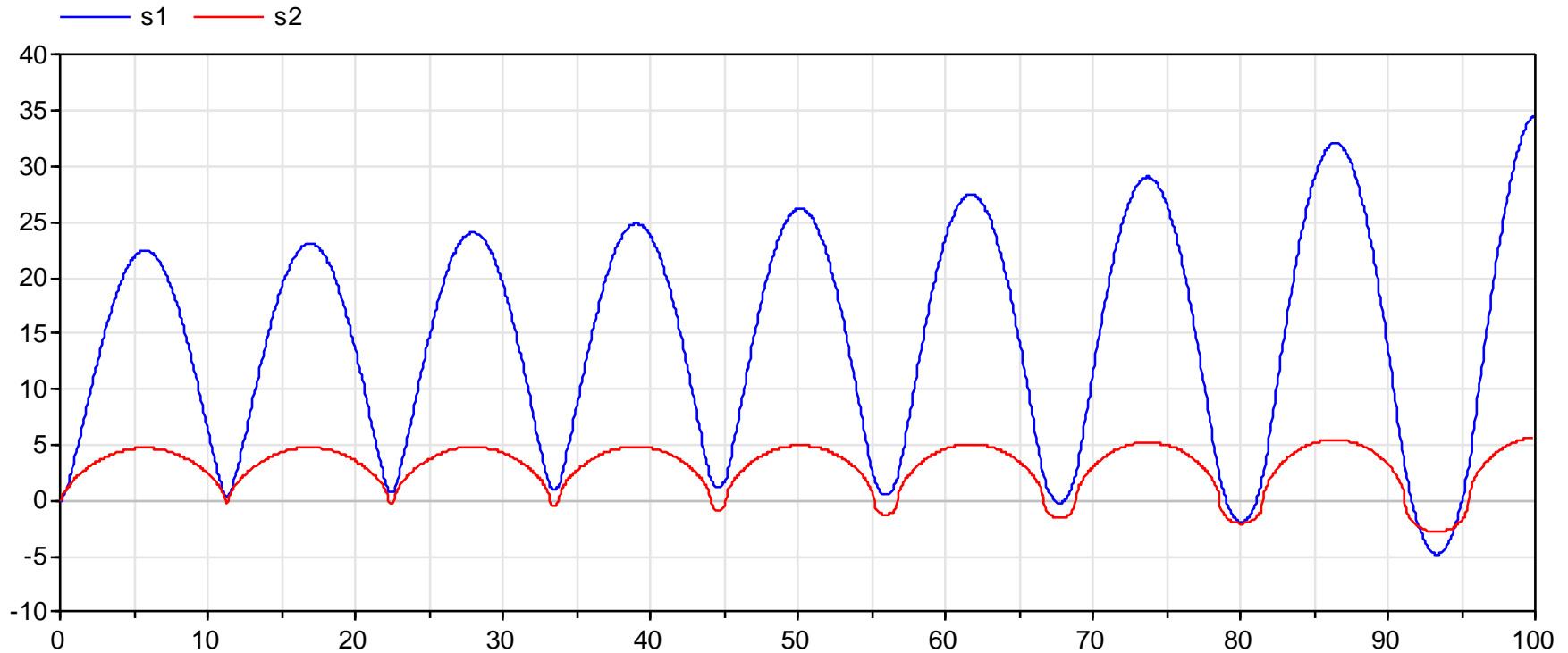
- But couldn't we formulate the same system, using just the velocities $\mathbf{v1}$ and $\mathbf{v2}$ instead of the positions $\mathbf{s1}$ and $\mathbf{s2}$?
- We could formulate $s1_int$ and $s2_int$ as integrals for $v1$ and $v2$.
- Here is why not:
(using DASSL with tolerance 0.01):

```
model TwoSpringsWithConstraint
  Real s1;
  Real s2;
  Real v1;
  Real v2;
  Real f;
  parameter Real m1 = 10;
  parameter Real m2 = 2;
  Real s1_int;
  Real s2_int;
  equation
    v1 = der(s1);
    v2 = der(s2);
    -1*s1 + f = m1*der(v1);
    -20*(s2-5) - f*abs(s2_int)*2
      = m2*der(v2);

    v1 = der(s1_int);
    v2 = der(s2_int);
    s1_int = abs(s2_int)*s2_int;
end TwoSpringsWithConstraint;
```

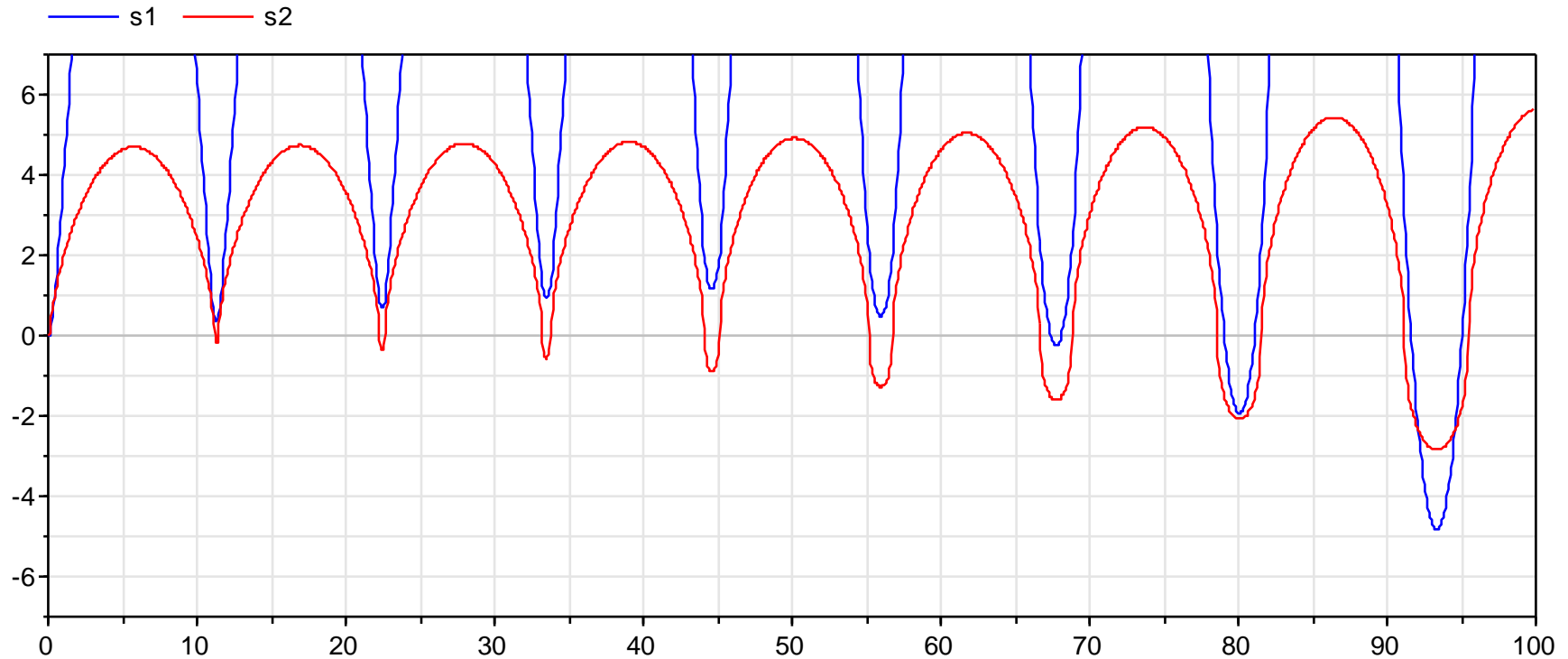
Holonomic Constraints: Example

- With DASSL and a local error tolerance of 0.01...



- ... the error has gotten worse and...

- With DASSL and a local error tolerance of 0.01...



- ...the holonomic constraint is lost.
- There is now a shift between the bodies.

- What has happened? Why does the system behave differently?
- Since s_1 and s_{1_int} are not algebraically coupled, they are separately integrated.
- The same holds for s_2 and s_{2_int} .
- Hence, the holonomic constraints becomes subject to an increasing numerical integration error.
- This can drastically change the systems behavior.

```
model TwoSpringsWithConstraint
  Real s1;
  Real s2;
  Real v1;
  Real v2;
  Real f;
  parameter Real m1 = 10;
  parameter Real m2 = 2;
  Real s1_int;
  Real s2_int;
  equation
    v1 = der(s1);
    v2 = der(s2);
    -1*s1 + f = m1*der(v1);
    -20*(s2-5) - f*abs(s2_int)*2
      = m2*der(v2);

    v1 = der(s1_int);
    v2 = der(s2_int);
    s1_int = abs(s2_int)*s2_int;
end TwoSpringsWithConstraint;
```

- What has happened? Why does the system behave differently?
- Since s_1 and s_{1_int} are not algebraically coupled, they are separately integrated.
- The same holds for s_2 and s_{2_int} .
- Hence, the holonomic constraints becomes subject to an increasing numerical integration error.
- This can drastically change the systems behavior.
- So... DON'T!

```
model TwoSpringsWithConstraint
  Real s1;
  Real s2;
  Real v1;
  Real v2;
  Real f;
  parameter Real m1 = 10;
  parameter Real m2 = 2;
  Real s1_int;
  Real s2_int;
  equation
    v1 = der(s1);
    v2 = der(s2);
    -1*s1 + f = m1*der(v1);
    -20*(s2-5) - f*abs(s2_int)*2
      = m2*der(v2);
v1 = der(s1_int);
v2 = der(s2_int);
    s1_int = abs(s2_int)*s2_int;
end TwoSpringsWithConstraint;
```

- For our mechanical components, this means that we have to use positions as potential variables:
- Each node was represented by a pair of variables

A **potential** variable

s (position for translational mechanics)

φ (angle for rotational mechanics)

and a **flow** variable

f (force for translational mechanics)

τ (force for rotational mechanics)

- We see that the new potential equations imply the old ones:

- **n-1 equalities**

Translational: $s_1 = s_2 = \dots = s_n$ implies $v_1 = v_2 = \dots = v_n$

Rotational: $\varphi_1 = \varphi_2 = \dots = \varphi_n$ implies $\omega_1 = \omega_2 = \dots = \omega_n$

- **1 balance equation**

Translational: $f_1 + f_2 + \dots + f_n = 0$

Rotational: $\tau_1 + \tau_2 + \dots + \tau_n = 0$

The information about the energy flow is still contained in our connector variables!

Now we can model the components: The dampers



$$\Delta v \cdot D = f$$

$$\Delta v = d(s_2 - s_1)/dt$$

$$f = f_2$$

$$0 = f_1 + f_2$$



$$\Delta \omega \cdot D = \tau$$

$$\Delta \omega = d(\varphi_2 - \varphi_1)/dt$$

$$\tau = \tau_2$$

$$0 = \tau_1 + \tau_2$$

Now we can model the components: The dampers



$$\Delta v \cdot D = f$$

This is totally fine

$$\Delta \omega \cdot D = \tau$$

$$\Delta v = d(s_2 - s_1) / dt$$

$$\Delta \omega = \text{der}(\varphi_2 - \varphi_1) / dt$$

$$f = f_2$$

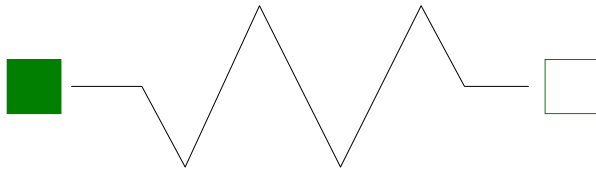
$$\tau = \tau_2$$

$$0 = f_1 + f_2$$

$$0 = \tau_1 + \tau_2$$

The derivatives are computed
symbolically not numerically

The springs: Since the new formulation is based on the positions, the model does not own a derivative anymore.

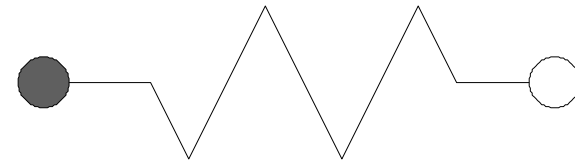


$$\Delta s \cdot C = f$$

$$\Delta s = (s_2 - s_1) - s_0$$

$$f = f_2$$

$$0 = f_1 + f_2$$



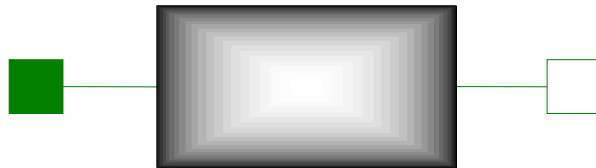
$$\Delta \varphi \cdot C = \tau$$

$$\Delta \varphi = (\varphi_2 - \varphi_1) - \varphi_0$$

$$\tau = \tau_2$$

$$0 = \tau_1 + \tau_2$$

Whereas the spring components have lost their integrator, the mass and inertia have gained one:

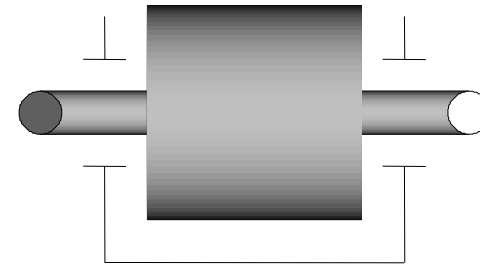


$$f = M \cdot dv/dt$$

$$v = ds_1/dt$$

$$s_2 = s_1$$

$$f = f_1 + f_2$$



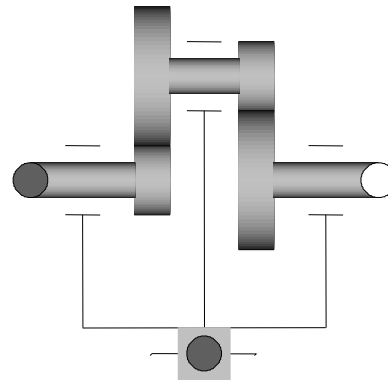
$$\tau = I \cdot d\omega/dt$$

$$\omega = d\varphi_1/dt$$

$$\varphi_2 = \varphi_1$$

$$\tau = \tau_1 + \tau_2$$

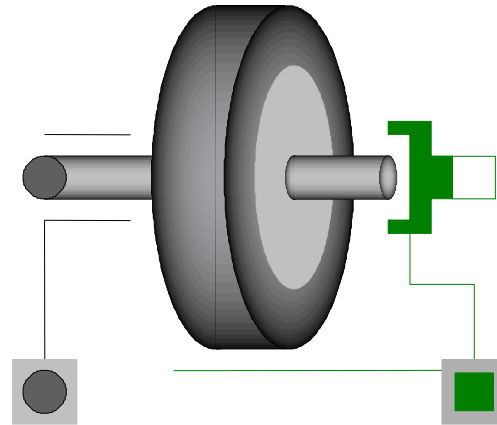
The transformer is represented by a gearbox. Its equation has hardly changed.



$$\varphi_2 = \text{Ratio} \cdot \varphi_1$$

$$\tau_1 = \text{Ratio} \cdot \tau_2$$

An ideal rolling wheel represents a transformation between translational and rotational movement.



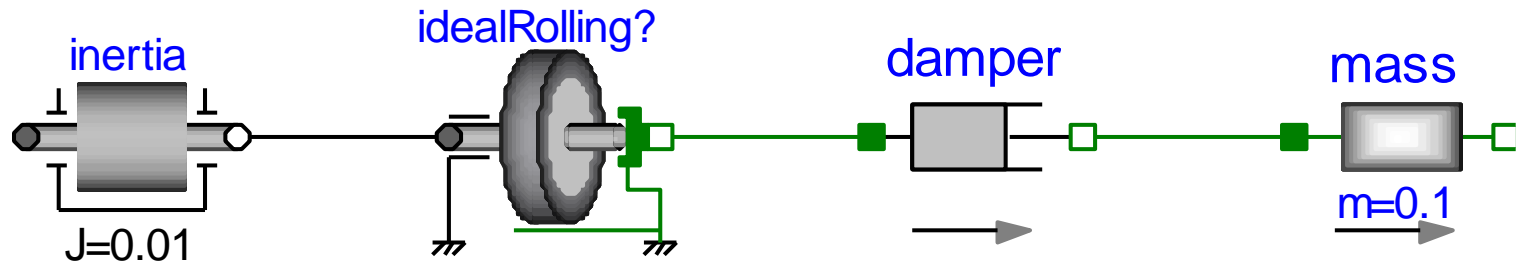
Ideal rolling means that the velocity of the virtual contact point is zero. The virtual contact point is located on the wheel.

$$\text{Radius} \cdot \varphi = s$$

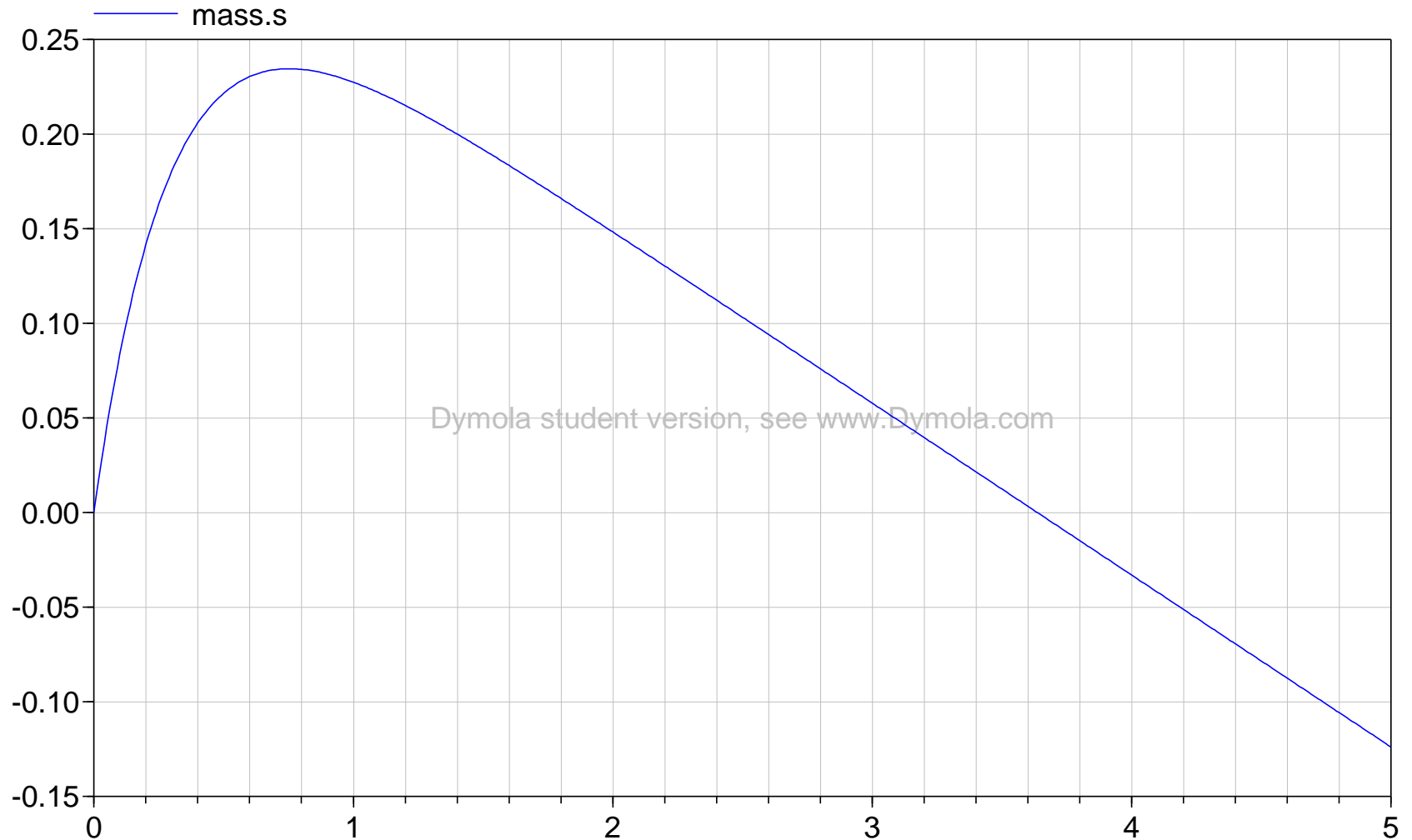
$$\tau = \text{Radius} \cdot f$$

Ball with counter spin

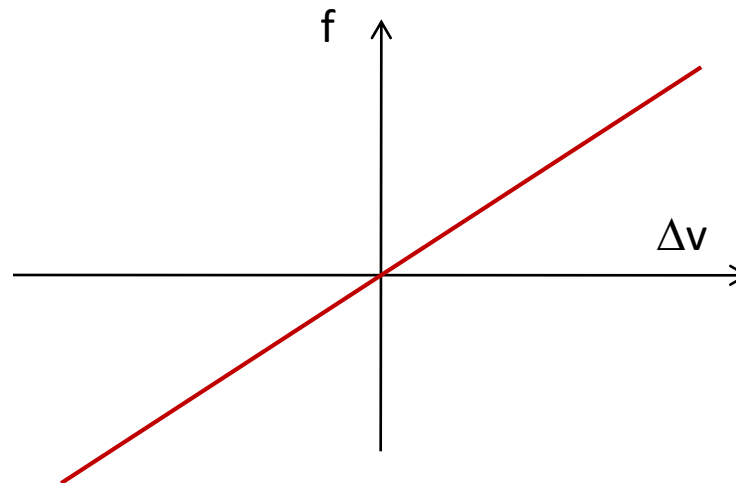
- Finally, let us model a simple mechanical system.
- A ball is placed on a table and propelled forwards with reverse spin. Eventually the spin will decelerate the ball and force him to roll backwards.
- Here is a first model of such a system.



Ball with counter spin

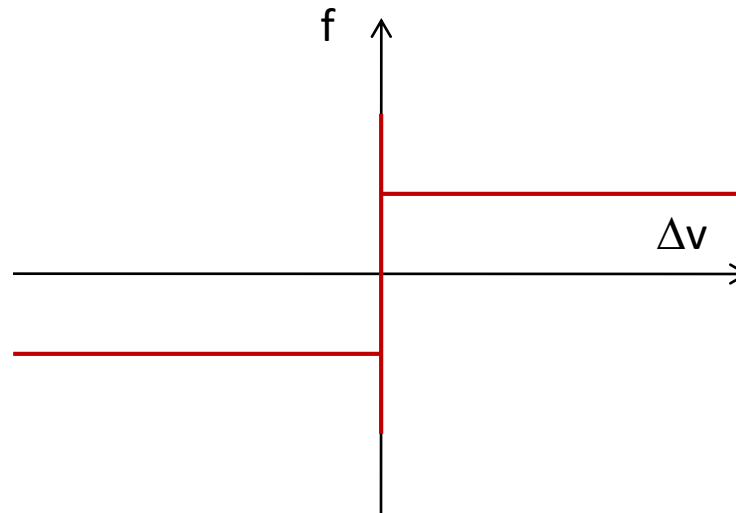


- The damper generates a friction force that is proportional to the difference in velocity.



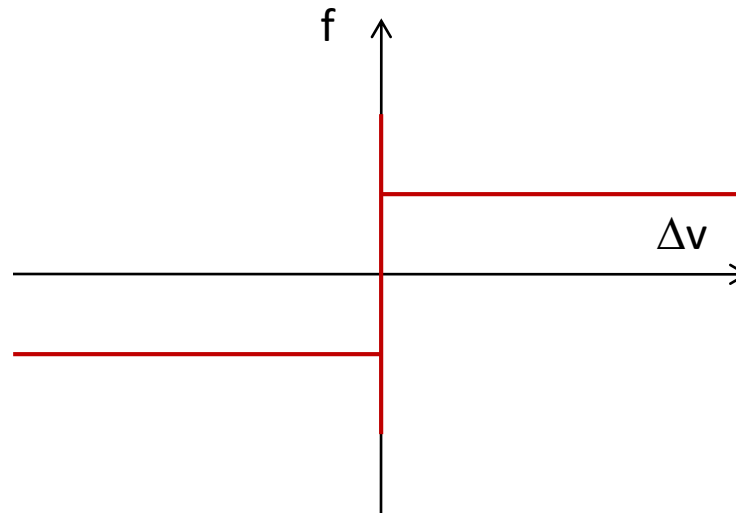
- The damper is not a good friction model. It is too “smooth”.
- Instead we want to use a dry friction model instead.

- The characteristic curve for dry friction is a multi-valued function and hence very tricky.



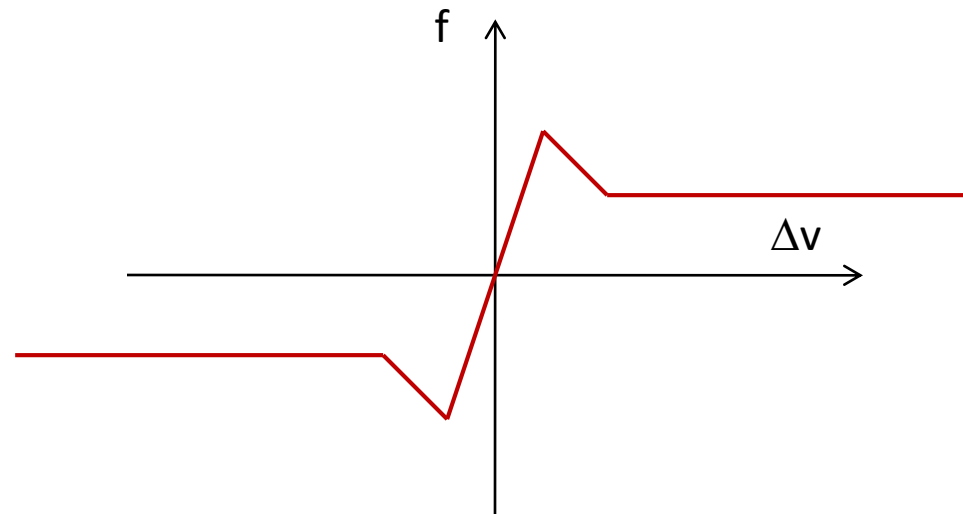
- The adhesive friction (“stiction”) is stronger than dry friction while sliding. The friction force always counteracts the movement.
- Hence, the curve contains discontinuities and represents infinite stiffness.
- The curve can also not be properly described by a mathematical function.

- Hence, the dry friction model of Modelica is pretty complicated and contains many language elements we do not know yet.



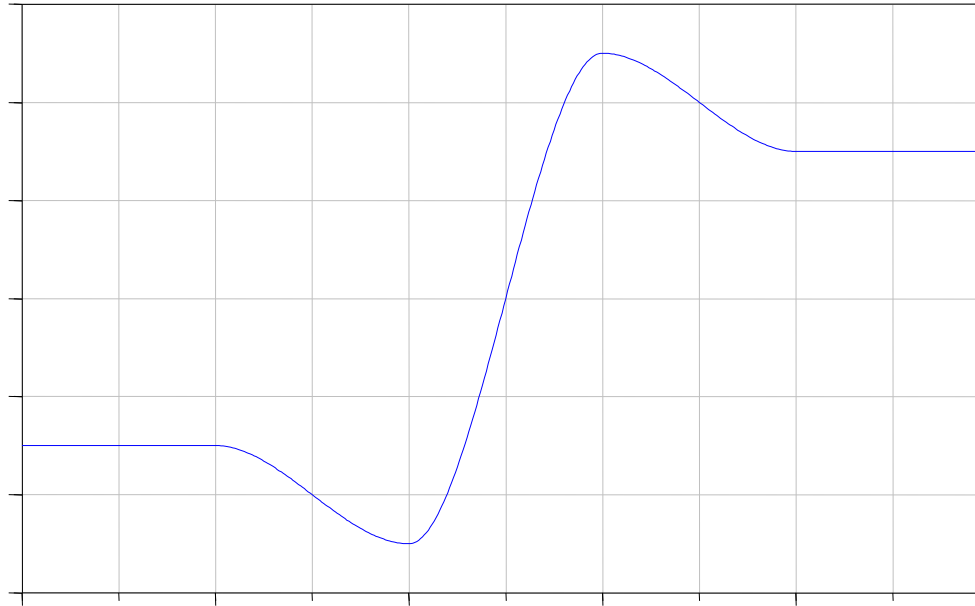
- Since we cannot cope with these discontinuities yet, we try to avoid them.
- We do so by regularizing the characteristic curve.

- To this end, we “stretch” the curve and transform it into a piecewise linear function.



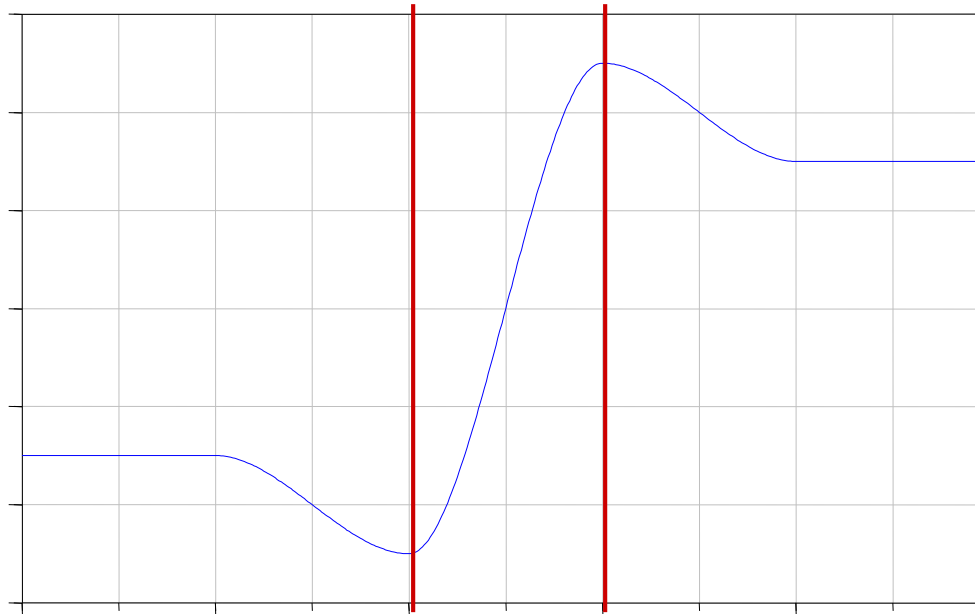
- The cost of this approach is: loss of precision and/or artificial stiffness.

- Instead of generating a piecewise linear function, we can also compose the function using three S-functions and two constant functions.



- The result is a nicely differentiable function.

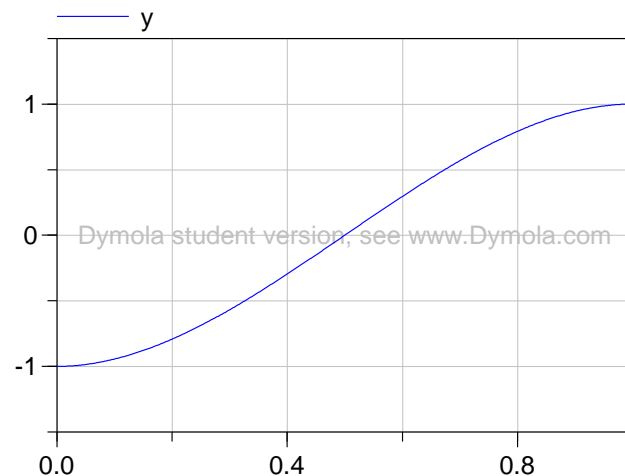
- Instead of generating a piecewise linear function, we can also compose the function using three S-functions.



- The result is a nicely differentiable function.

- For the S-Function, we use a polynomial:

$$y = -x^3/2 + 3x/2$$



```
function S_Func
```

```
  input Real x;
```

```
  output Real y;
```

```
algorithm
```

```
  if x > 1 then
```

```
    y := 1;
```

```
  elseif x < -1 then
```

```
    y := -1;
```

```
  else
```

```
    y := -0.5*x^3 + 1.5*x;
```

```
  end if;
```

```
end S_Func;
```

- For the S-Function, we use a polynomial:

$$y = -x^3/2 + 3x/2$$

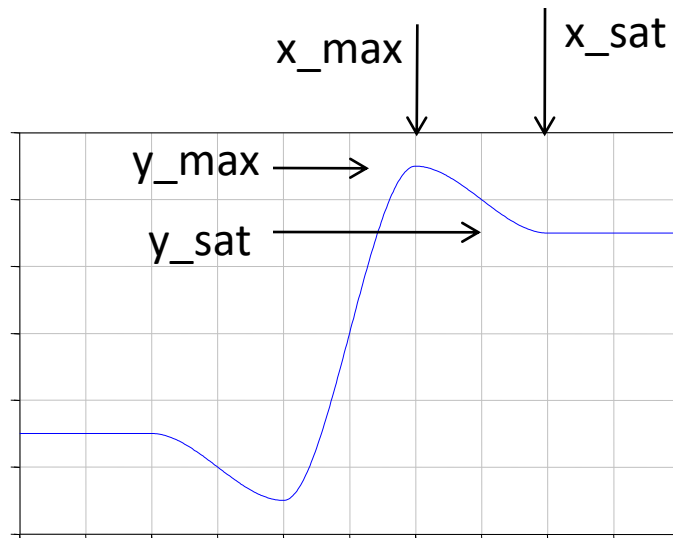
- Then, we provide inputs in order to scale the function to fit an arbitrary rectangle

$(x_{\min}, y_{\min}, x_{\max}, y_{\max})$

- The annotation tells Dymola that the function is differentiable once. So there are no discontinuities.
- This is important for the ODE-solver.

```
function S_Func "Models an S-Function"  
  input Real x_min;  
  input Real x_max;  
  input Real y_min;  
  input Real y_max;  
  input Real x;  
  output Real y;  
  
  protected  
    Real x2;  
  
  algorithm  
    x2 := x - x_max/2 - x_min/2;  
    x2 := x2*2/(x_max-x_min);  
    if x2 > 1 then  
      y := 1;  
    elseif x2 < -1 then  
      y := -1;  
    else  
      y := -0.5*x2^3 + 1.5*x2;  
    end if;  
    y := y*(y_max-y_min)/2;  
    y := y + y_max/2 + y_min/2;  
    annotation(smoothOrder=1);  
  end S_Func;
```

- We may use the S-Function in order to compose the point-symmetric Triple S-Function:



```
function TripleS_Func
```

```
input Real x_max;  
input Real x_sat;  
input Real y_max;  
input Real y_sat;
```

```
input Real x;  
output Real y;
```

```
algorithm
```

```
if x > x_max then
```

```
  y := S_Func(x_max, x_sat,  
              y_max, y_sat, x);
```

```
elseif x < -x_max then
```

```
  y := S_Func(-x_max, -x_sat,  
              -y_max, -y_sat, x);
```

```
else
```

```
  y := S_Func(-x_max, x_max, -  
              y_max, y_max, x);
```

```
end if;
```

```
  annotation(smoothOrder=1);
```

```
end TripleS_Func;
```

- Now we can model our own dry friction component:



```
model DryFriction
  extends Modelica.Mechanics.
    Translational.Interfaces.
    PartialCompliantWithRelativeStates;

  import SI = Modelica.SIunits;

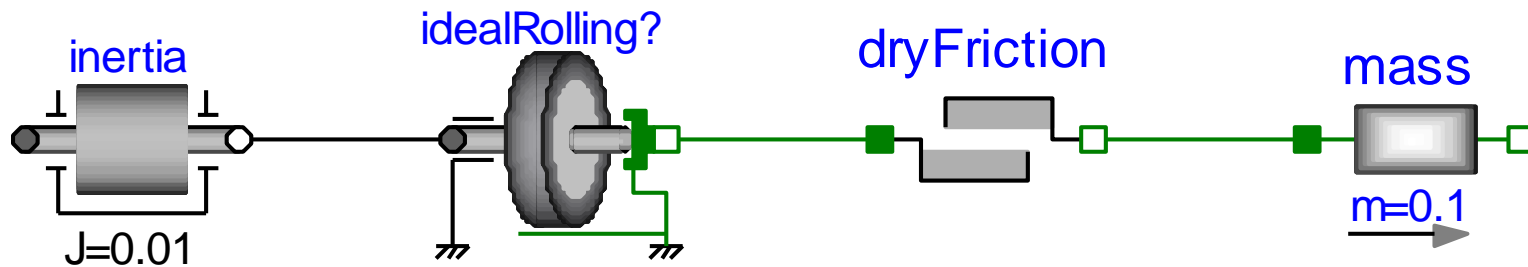
  parameter SI.Force N
    "normal force";
  parameter SI.Velocity vAdhesion
    "adhesion velocity";
  parameter SI.Velocity vSlide
    "sliding velocity";
  parameter Real mu_A
    "friction coefficient at adhesion";
  parameter Real mu_S
    "friction coefficient at sliding";

  equation
    f = N*TripleS_Func(vAdhesion,vSlide,
                      mu_A,mu_S,v_rel);

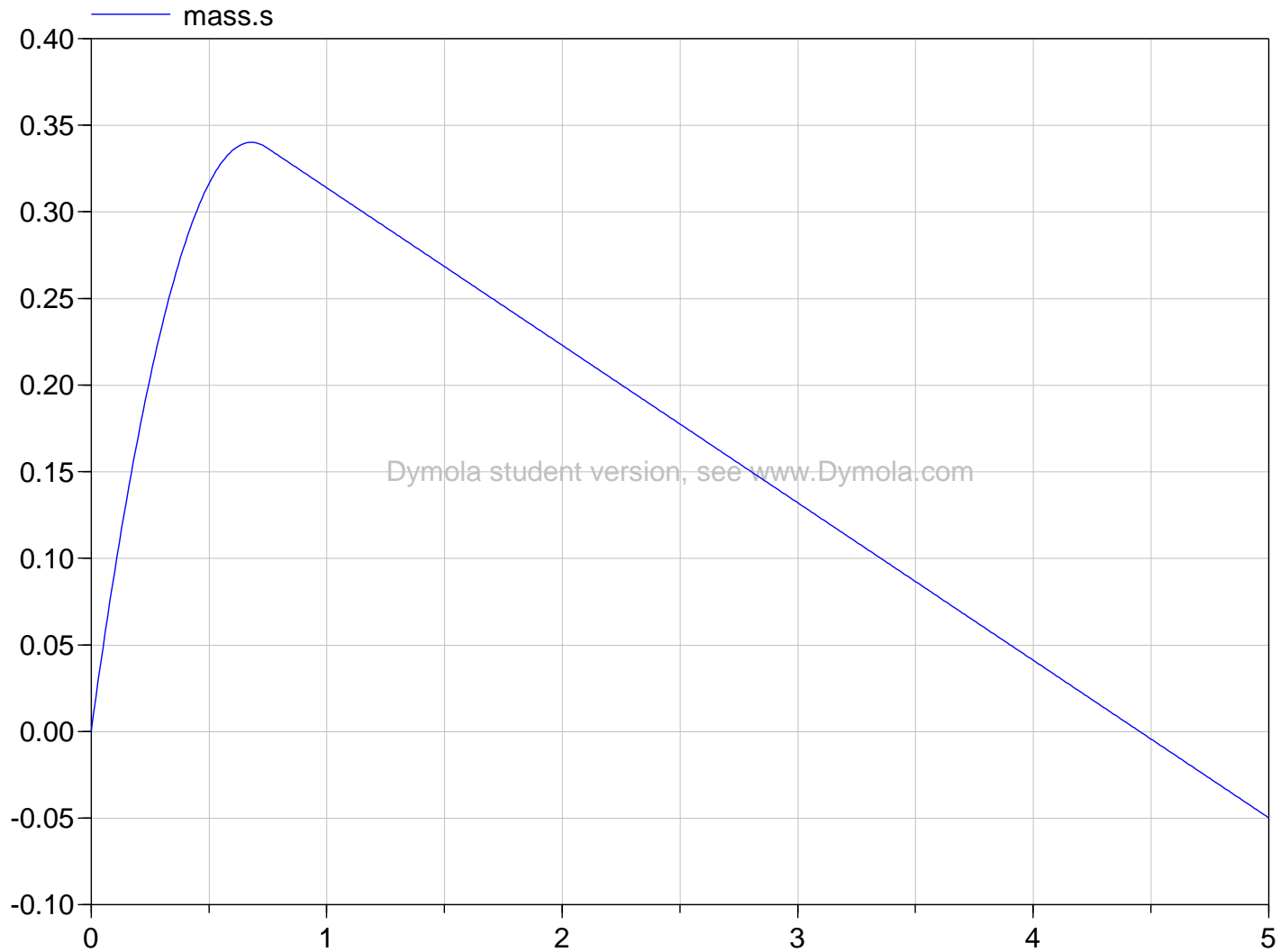
end DryFriction;
```

Counter spin and dry friction

- Here is the application of our dry-friction component.



Counter spin and dry friction



- Rotational and translational mechanics can be treated the same way.
- The proper formulation of mechanical systems requires the formulation of holonomic constraints.
- In order to enable this, positions and not velocities form the potential connector variables.
- Consequently, the derivatives are redistributed within the components.
- We learnt about dry friction and regularization.

Questions ?