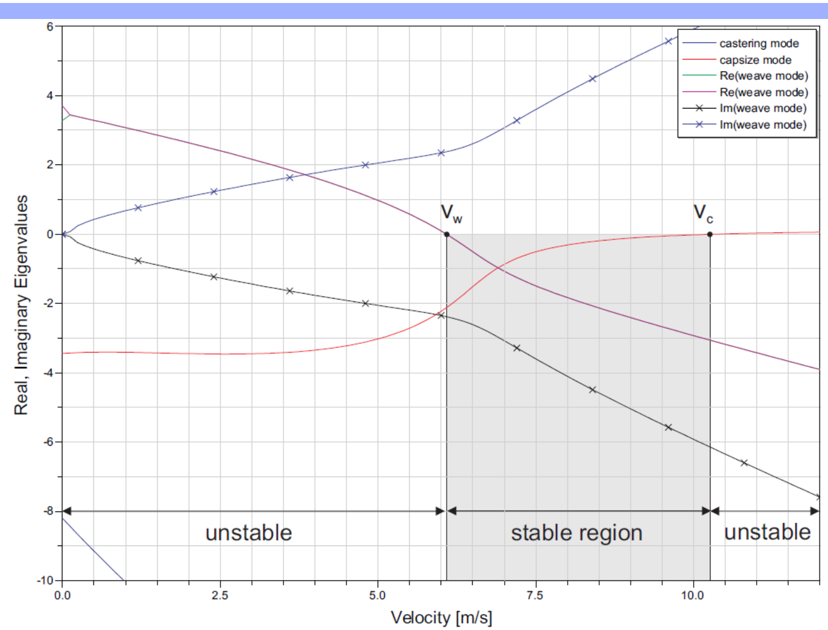
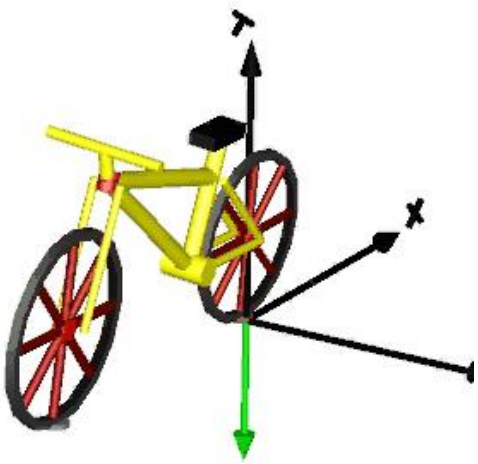
Virtual Physics
Equation-Based Modeling

TUM, January 10, 2023

Higher Level Modeling Tasks: Stability Analysis





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Recap: State-Space Form

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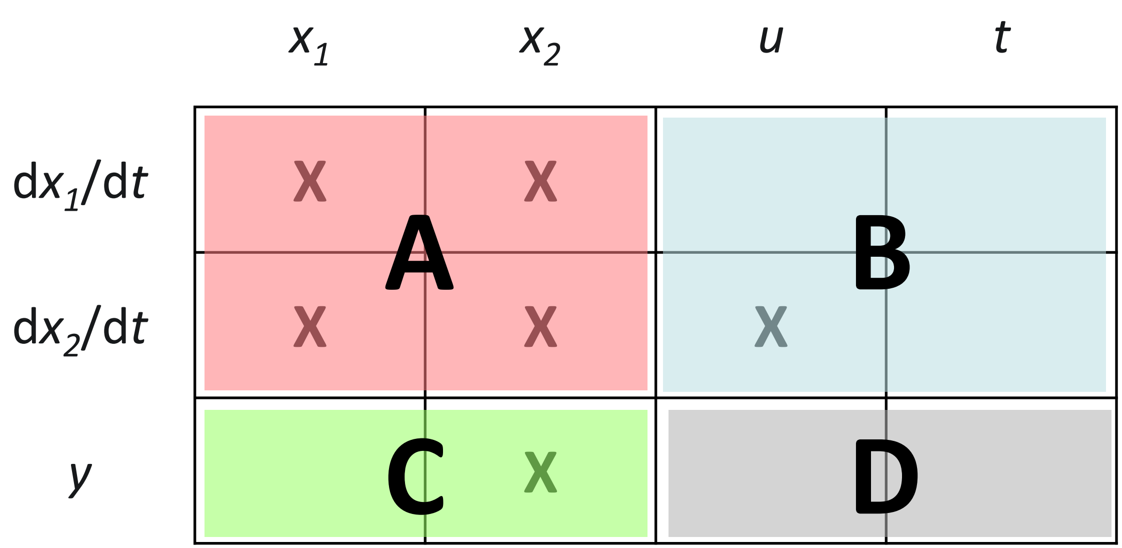
- We have learned that for linear systems, the state-space form can be described by four sub-matrices **A**, **B**, **C**, and **D**.

$$d\mathbf{x}/dt = \mathbf{Ax} + \mathbf{B}(\mathbf{u}, t)$$

$$\mathbf{y} = \mathbf{Cx} + \mathbf{D}(\mathbf{u}, t)$$



Recap: State-Space Form

- For the actual dynamics, the matrix **A** is of primary interest.

$$d\mathbf{x}/dt = \mathbf{A}\mathbf{x}$$

	x_1	x_2	u	t
dx_1/dt	X	X		
dx_2/dt	X	X	X	
y		X		

A

Stability in 1D Systems

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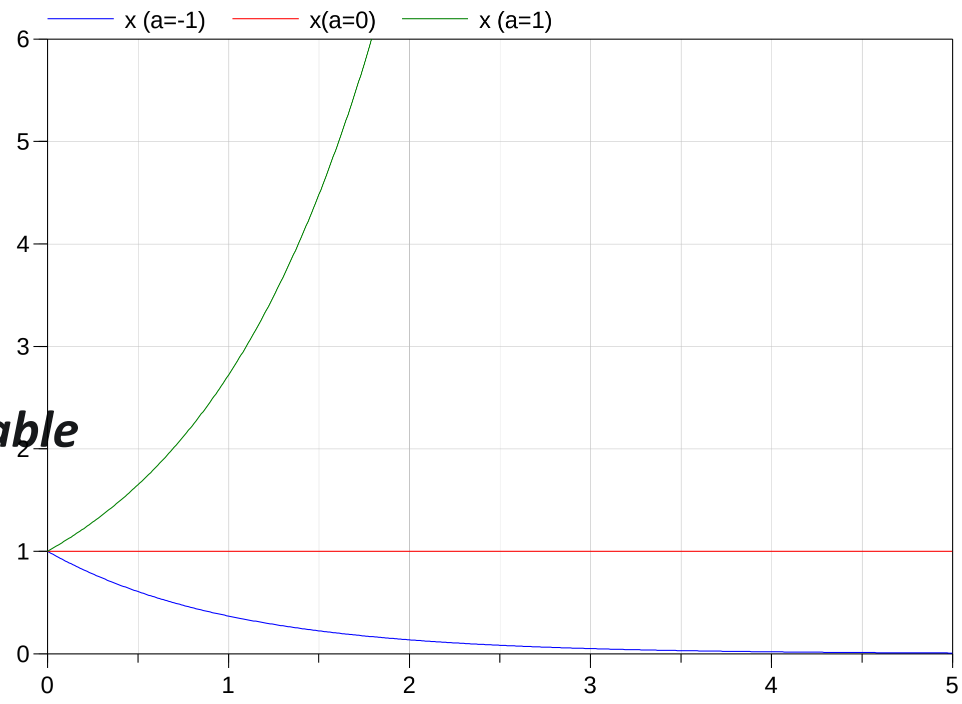
Let us look at the general one-dimensional linear system

$$dx/dt = ax \text{ (with } x_{start} = 1\text{)}$$

Definition:

- a > 0:
  The system is ***unstable***
- a < 0:
  The system is **asymptotically *stable***
- a = 0:
  The system is ***marginally stable***



Stability in *n*D Systems

What about the multi-dimensional case?

$$d\mathbf{x}/dt = \mathbf{A}\mathbf{x}$$

- We can perform an eigenvalue decomposition:

  $$\mathbf{A} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^{-1}$$

  where **Q** consists in the eigenvectors and **Λ** is a diagonal matrix containing the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$.

- The system is ***asymptotically stable*** iff all eigenvalues are smaller than 0.

- The system is ***marginaly stable*** if all eigenvalues are smaller or equal than 0 and at least one eigenvaule is 0.

- The system is ***unstable*** otherwise.

- For complex eigenvalues, only the real part is of concern.

Stability in *n*D Systems

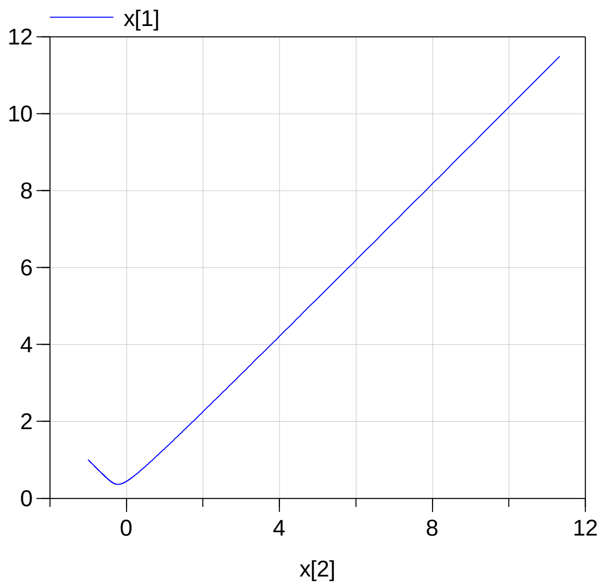
Example:

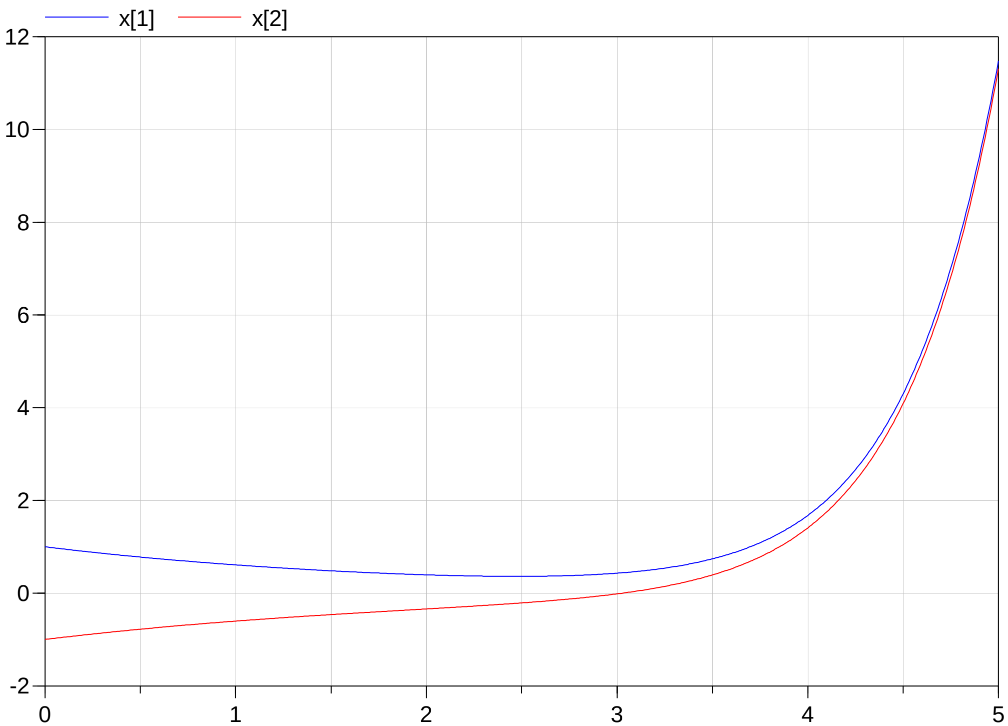
$$\mathbf{A} = \begin{pmatrix} 3/4 & 5/4 \\ 5/4 & 3/4 \end{pmatrix}$$

- We can perform an eigenvalue decomposition:

$$\mathbf{A} = \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & -0.5 \end{pmatrix} \begin{pmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}$$

The largest eigenvalue is 2.
Hence the system is unstable.





Stability in *n*D Systems

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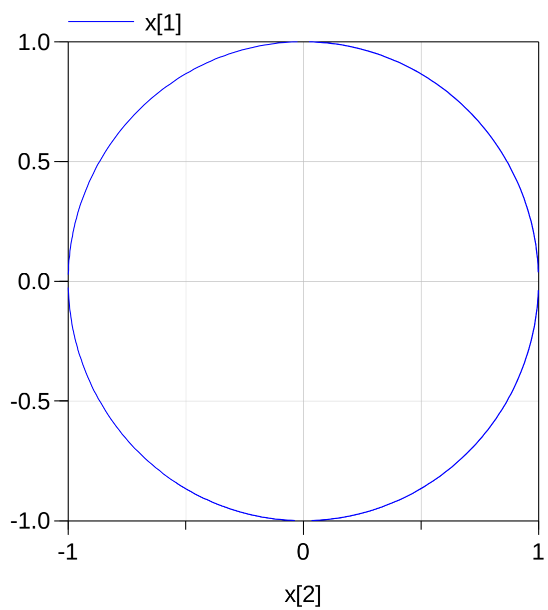
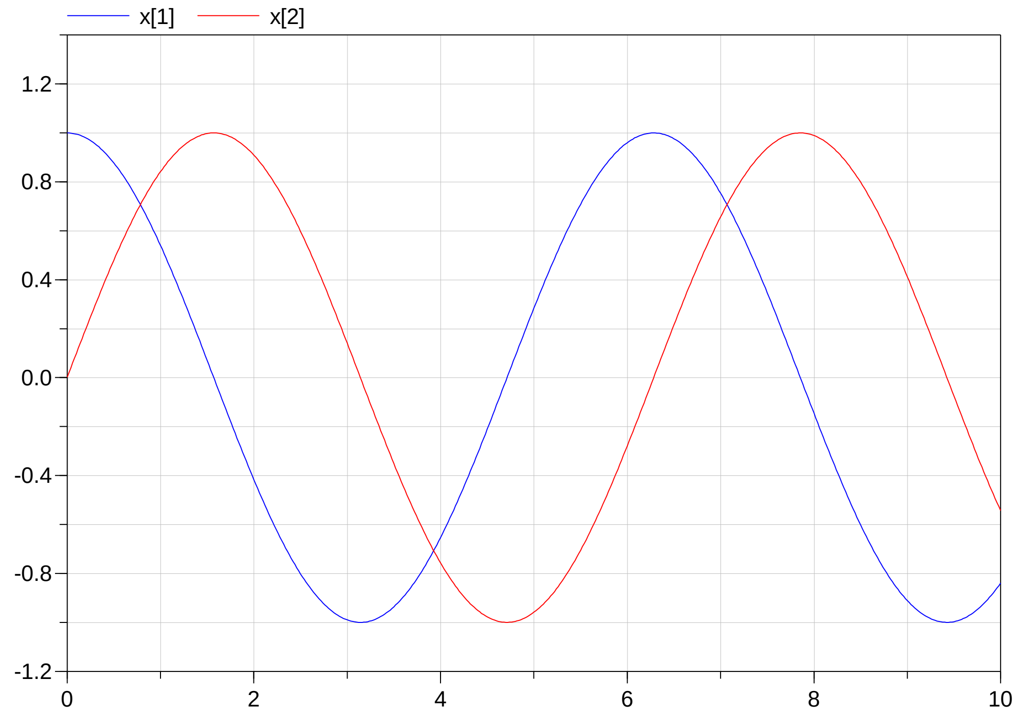
Example:

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

- This matrix has the complex-conjugate pair of eigenvalues: (i, -i)

- The real part of all eigenvalues is zero: Hence the system is marginally stable.





Stability in *n*D Systems

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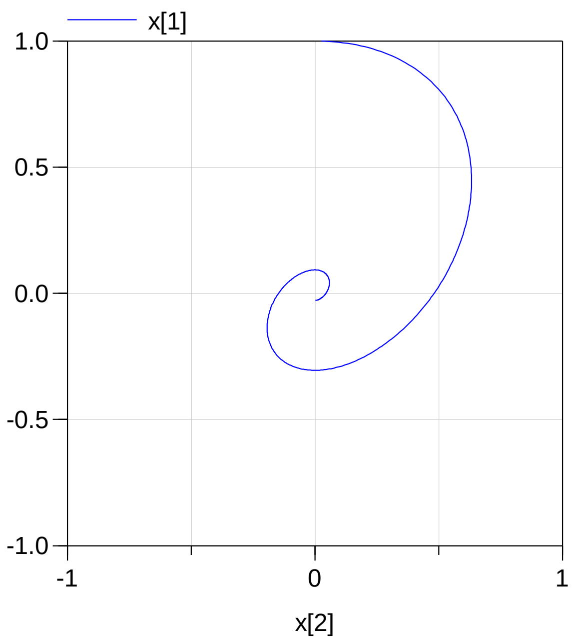
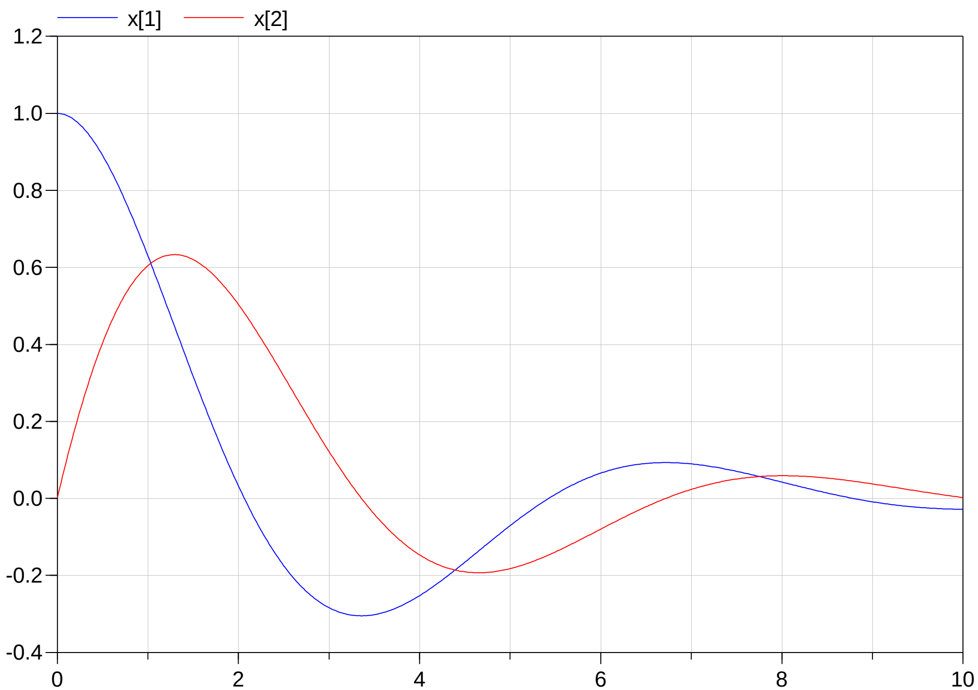
Example:

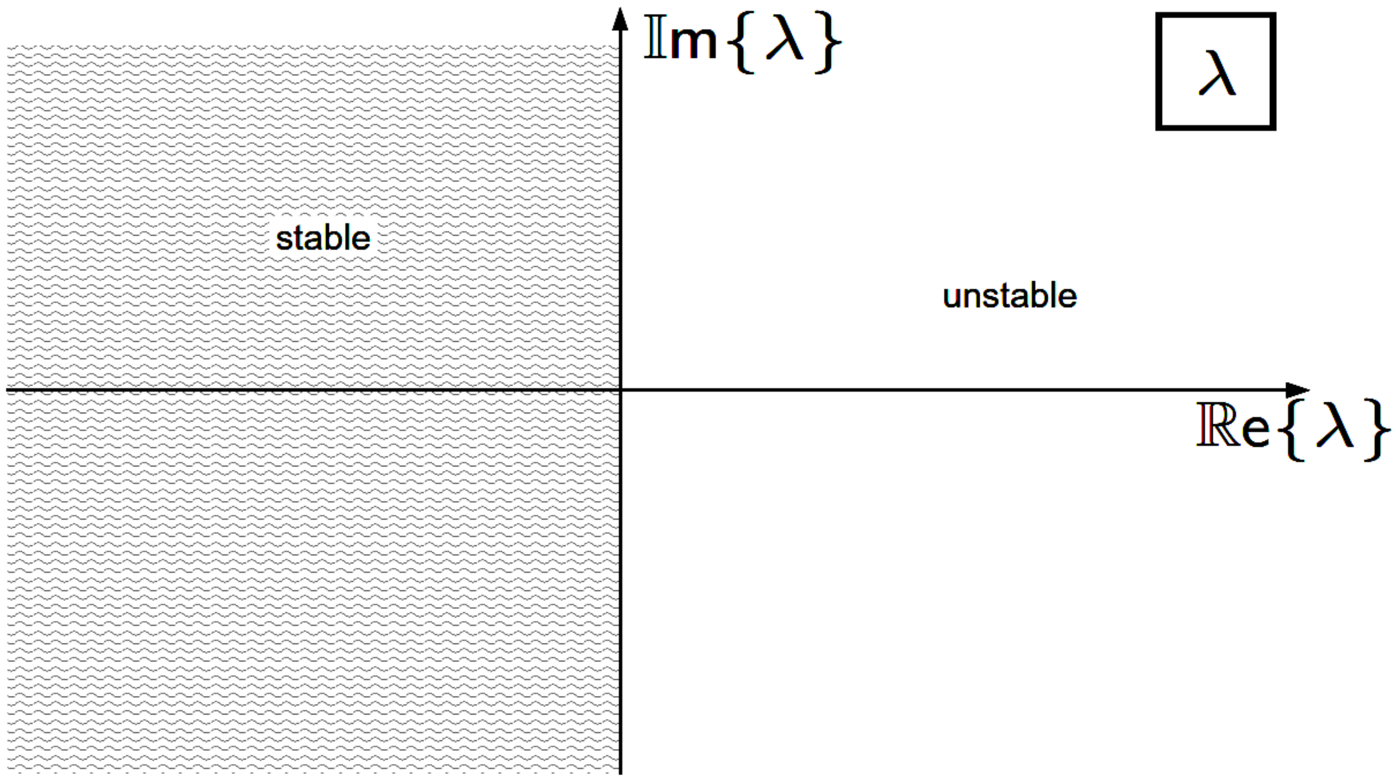
$$A = \begin{pmatrix} 0 & 1 \\ -1 & -\sqrt{2} \end{pmatrix}$$

- This matrix has the complex-conjugate pair of eigenvalues: $(-\sqrt{2}/2+ \sqrt{2}/2i, -\sqrt{2}/2- \sqrt{2}/2i\ )$
- The real part of all eigenvalues is smaller than zero: Hence the system is stable.





Domain of Analytical Stability



Non-linear systems

Most of our systems have been non-linear. What can we say about them?

- For non-linear systems (in general), a global statement on the stability cannot be given.
- Indeed, a non-linear system may be stable at one point in state-space and unstable at another point.
- However, for a potential equilibrium point in state-space we can examine the stability locally by performing a linearization of the system.

Linearization

Let us perform a linearization:

- Given the non-linear system in state-space form:

$$d\mathbf{x}/dt = f(\mathbf{x},\mathbf{u},t)$$

- We can approximate this system around the point ($\mathbf{x}_p$, $\mathbf{u}_p$, t_p) by

$$d\mathbf{x}/dt = \mathbf{A}\Delta\mathbf{x} + \mathbf{B}\,(\Delta\mathbf{u}, \Delta t) + f(\mathbf{x}_p, \mathbf{u}_p, t_p)$$

- where **A** is formed by the Jacobian of f() at ($\mathbf{x}_p$, $\mathbf{u}_p$, t_p) :
  (and the same is done to get B)

$$\mathbf{A} = \begin{bmatrix} \partial f_1/\partial \mathbf{x}_1 & \partial f_1/\partial \mathbf{x}_2 & \cdots & \partial f_1/\partial \mathbf{x}_n \\ \partial f_2/\partial \mathbf{x}_1 & \partial f_2/\partial \mathbf{x}_2 & \cdots & \partial f_2/\partial \mathbf{x}_n \\ \cdots & \cdots & \cdots & \cdots \\ \partial f_n/\partial \mathbf{x}_1 & \partial f_n/\partial \mathbf{x}_2 & \cdots & \partial f_n/\partial \mathbf{x}_n \end{bmatrix}$$

Linearization of a Pendulum

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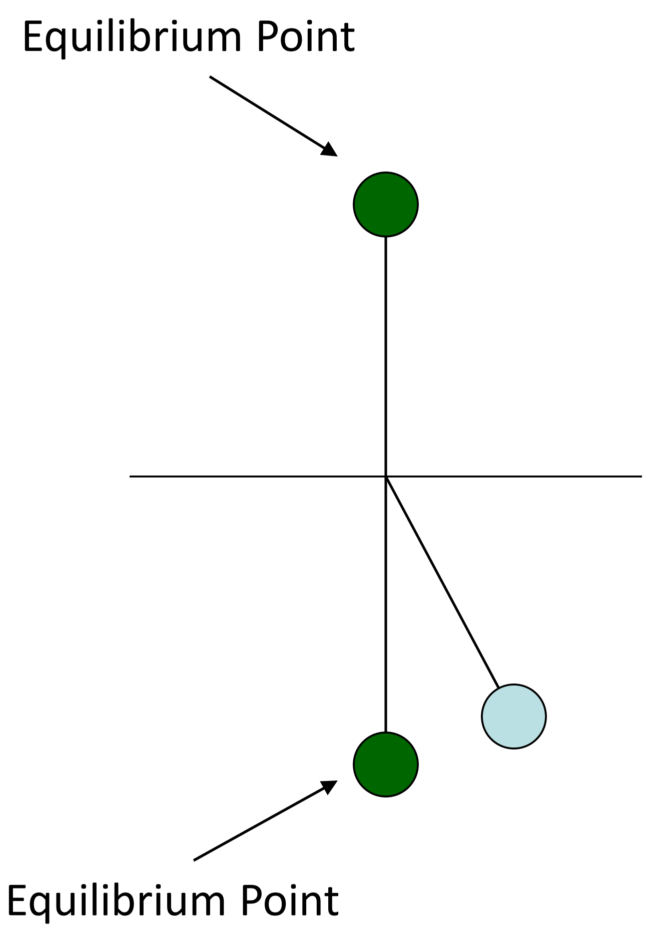
Let us look at a simple example: The pendulum

- It has two potential points of equilibrium:
  The lower position ($\varphi = 0$)
  The upright position ($\varphi = \pi$)

- The pendulum can be described by a matrix with non-linear elements:

$d\varphi/dt$	=	0	+	ω
$d\omega/dt$	=	$-\sin(\varphi)\cdot g/l$	+	$-\mu_D/(m\cdot l^2)\ \omega$



Linearization of a Pendulum

Let us linearize the system at its lower equilibrium ($\varphi = 0$):

$$\begin{bmatrix} d\varphi/dt \\ d\omega/dt \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -g/l & -\mu_D/(m \cdot l^2) \end{bmatrix} \begin{bmatrix} \varphi \\ \omega \end{bmatrix}$$

- Or simply...

$$\begin{bmatrix} d\varphi/dt \\ d\omega/dt \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -c & -b \end{bmatrix} \begin{bmatrix} \varphi \\ \omega \end{bmatrix}$$

with $b > 0$ and $c > 0$.

Linearization of a Pendulum

To compute the eigenvalues, we derive the determinant of:

$$\begin{bmatrix} 0-\lambda & 1 \\ -c & -b-\lambda \end{bmatrix}$$

- This yields the characteristic polynomial whose roots are the eigenvalues:

$$\lambda^2 + b\lambda + c$$

- The roots are located at:

$$-b/2 + \tfrac{1}{2}\sqrt{(b^2 - 4c)} \quad \text{and} \quad -b/2 - \tfrac{1}{2}\sqrt{(b^2 - 4c)}$$

- Since $b>0$ and $c>0$, the roots will be negative in their real parts.

- This equilibrium point is a stable equilibrium

Linearization of a Pendulum

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When we linearize the pendulum in upright position ($\varphi = \pi$), there occurs a switch in sign.

- The coefficient $c = (\sin(\varphi)\cdot g/l)/d\varphi$ becomes negative:

$$c = -g/l$$

- The roots are still located at:

$$-b/2 + \tfrac{1}{2}\sqrt{(b^2 - 4c)} \quad \text{and} \quad -b/2 - \tfrac{1}{2}\sqrt{(b^2 - 4c)}$$

- With $c < 0$, the discriminant is positive. This means that both roots are real values. Also the root $-b/2 + \tfrac{1}{2}\sqrt{(b^2 - 4c)}$ will be positive.

- This equilibrium point is unstable.

Stability Analysis

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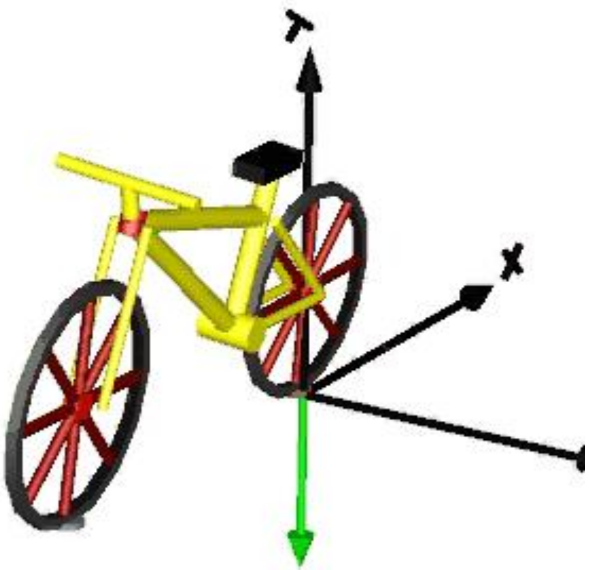
Let us analyze the stability of an uncontrolled bicycle:

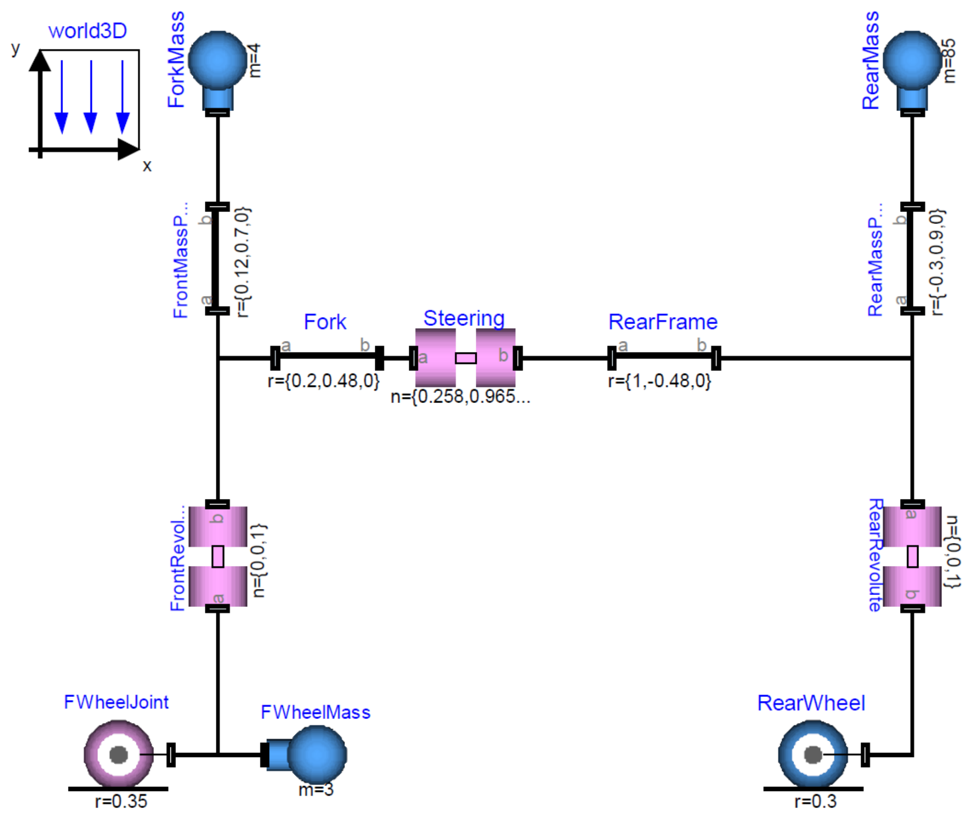
- An ideal rolling bicycle represents a non-linear system with non-holonomic constraints
- The bicycle is known to be (self-) stable within a certain velocity range.
- We want to determine this velocity range.



Stability Analysis

First, we create a simple model of 3D- bicycle

- The model is built out of the usual components.
- There are special models for the ideal rolling wheels.
- The driver is included in this model and considered to be “rigid”.



Stability Analysis

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Let us look at the state-space vector **x**. In total, we have 10 state variables.

- 7 on the level of position (1 Free Body (1x6) + 3 revolute joints (3x1) – 2 holonomic contraints for placing the wheels (2) = 7)
- 3 on the level of velocity (1 Free Body (1x6) + 3 revolute joints (3x1) – 2 holonomic contraints for placing the wheels (2) – 4 non-holonomic constraints at the wheel = 3)

$$x = \begin{pmatrix} \texttt{SteeringRevolute.}\varphi \\ \texttt{FWRev.}\varphi \\ \texttt{RearWheel.}rx_0 \\ \texttt{RearWheel.}ry_0 \\ \texttt{RearWheel.}\psi \\ \texttt{RearWheel.}\varphi \\ \texttt{RearWheel.}\theta \\ \texttt{RearWheel.}\dot{\psi} \\ \texttt{RearWheel.}\dot{\varphi} \\ \texttt{RearWheel.}\dot{\theta} \end{pmatrix} \begin{matrix} \texttt{steering angle} \\ \texttt{front wheel revolute angle} \\ \texttt{rear wheel's x-position} \\ \texttt{rear wheel's y-position} \\ \texttt{rear wheel's orientation angle} \\ \texttt{rear wheel's lean angle} \\ \texttt{rear wheel's roll angle} \\ \texttt{rear wheel's orientation a. vel.} \\ \texttt{rear wheel's lean a. vel.} \\ \texttt{rear wheel's roll velocity} \end{matrix}$$

Stability Analysis

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Let us look at the linearization matrix **A**

- Dymola is able to perform a linearization automatically. This saves a great deal of work.
- The linearization point is an upright position with a given driving velocity.
- Here is a typical example of **A:**

$$A = \begin{pmatrix} -62.5000 & 0 & 0 & 0 & 0 & 0 & 0 & -13.2638 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.8571 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.3000 \\ 0 & 0 & 0 & 0 & -5.0000 & 0 & 0 & 0 & -0.3000 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.0000 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.0000 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.0000 \\ 225.0401 & 0 & 0 & 0 & 0 & 0.8388 & 0 & 47.4538 & 1.5245 & 0 \\ -83.1329 & 0 & 0 & 0 & 0 & 9.5182 & 0 & -22.5493 & -0.5608 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Stability Analysis

The complete system is obviously unstable (after all, the bicycle is driving…)

- We are only interested to analyze the subsystem that is relevant for the driving dynamics.
- Hence we select the following states:
  Steering angle, lean angle, orientation- and lean-angle velocity.

$x =$

SteeringRevolute.φ	steering angle
FWRev.φ	front wheel revolute angle
RearWheel.rx_0	rear wheel's x-position
RearWheel.ry_0	rear wheel's y-position
RearWheel.ψ	rear wheel's orientation angle
RearWheel.φ	rear wheel's lean angle
RearWheel.θ	rear wheel's roll angle
RearWheel.$\dot{\psi}$	rear wheel's orientation a. vel.
RearWheel.$\dot{\varphi}$	rear wheel's lean a. vel.
RearWheel.$\dot{\theta}$	rear wheel's roll velocity

Stability Analysis

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This is only valid, if the driving dynamics are not influenced by the other states.

- For the remaining 5 positional states this is blatantly evident.
- It does not matter where the bicycle is placed or in what direction it points. The roll angle of the wheels is also unimportant

$$x = \begin{pmatrix} \texttt{SteeringRevolute.}\varphi \\ \texttt{FWRev.}\varphi \\ \texttt{RearWheel.}rx_0 \\ \texttt{RearWheel.}ry_0 \\ \texttt{RearWheel.}\psi \\ \texttt{RearWheel.}\varphi \\ \texttt{RearWheel.}\theta \\ \texttt{RearWheel.}\dot{\psi} \\ \texttt{RearWheel.}\dot{\varphi} \\ \texttt{RearWheel.}\dot{\theta} \end{pmatrix} \quad \begin{matrix} \texttt{steering angle} \\ \texttt{front wheel revolute angle} \\ \texttt{rear wheel's x-position} \\ \texttt{rear wheel's y-position} \\ \texttt{rear wheel's orientation angle} \\ \texttt{rear wheel's lean angle} \\ \texttt{rear wheel's roll angle} \\ \texttt{rear wheel's orientation a. vel.} \\ \texttt{rear wheel's lean a. vel.} \\ \texttt{rear wheel's roll velocity} \end{matrix}$$

Stability Analysis

Things are tricky for the rolling velocity of the rear wheel (driving velocity) It obviously influences the dynamic behavior.

- Although we exclude this state here, it is actually part of the potentially stable sub-system.
- All 5 states are sufficient to describe the potential and kinetic energy of the system. Hence the rolling velocity cannot depart from the equilibrium point when all other states do not. But only if the system is non-dissipative and the total amount of energy is conserved.
- And indeed, any kind of friction (dissipation) turns the whole bicycle into an unstable system. The bicycle will slow down and fall over.

$x =$

RearWheel.φ	rear wheel's lean angle
RearWheel.θ	rear wheel's roll angle
RearWheel.$\dot{\psi}$	rear wheel's orientation a. vel.
RearWheel.$\dot{\varphi}$	rear wheel's lean a. vel.
RearWheel.$\dot{\theta}$	rear wheel's roll velocity

Stability Analysis

Now we can reduce the stability analysis to a 4D subpart.

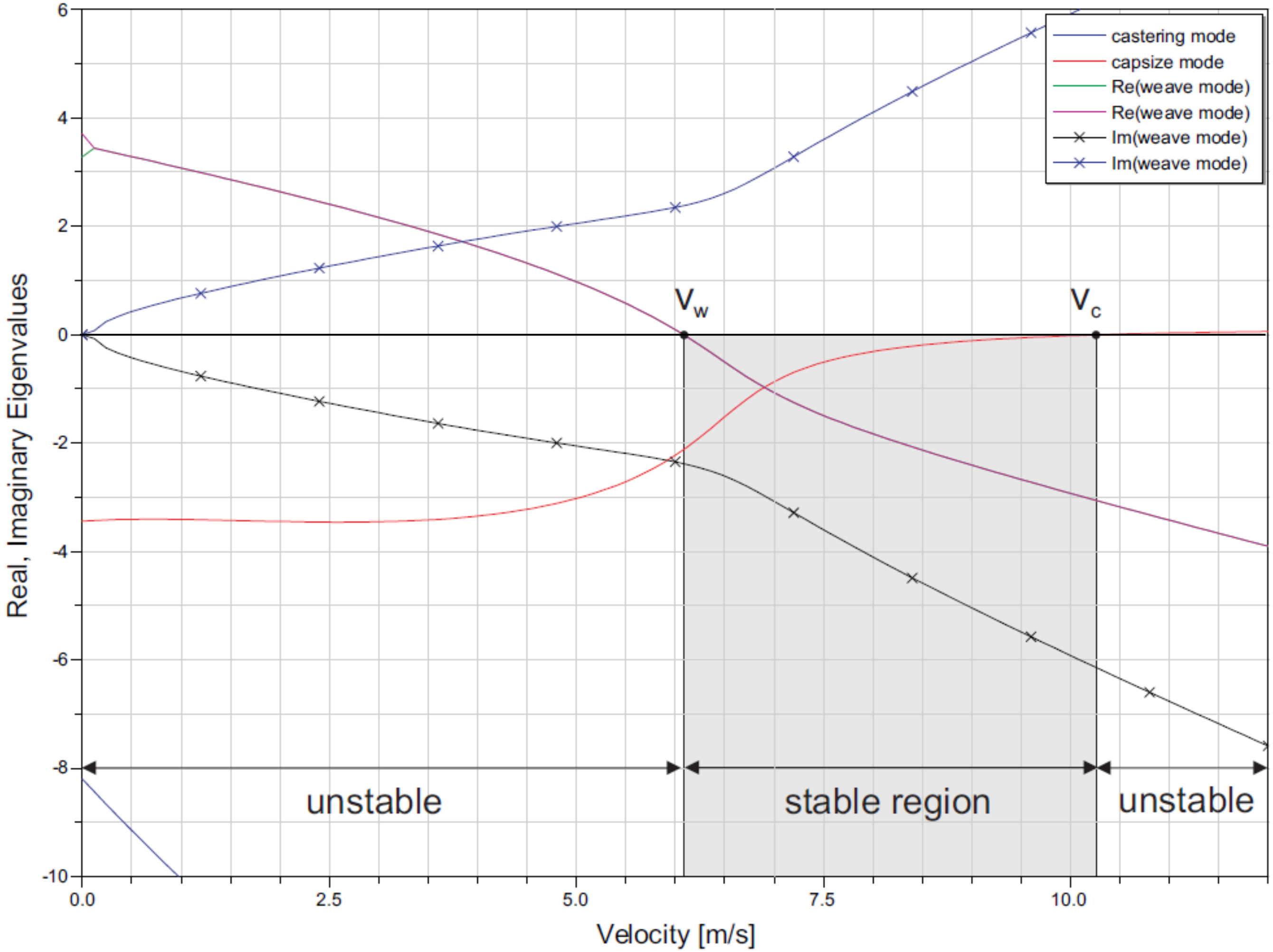
$$x_{stab} = \begin{pmatrix} \texttt{SteeringRevolute.}\varphi \\ \texttt{RearWheel.}\varphi \\ \texttt{RearWheel.}\dot{\psi} \\ \texttt{RearWheel.}\dot{\varphi} \end{pmatrix} \quad \begin{matrix} \texttt{steering angle} \\ \texttt{rear wheel's lean angle} \\ \texttt{rear wheel's orientation a. vel.} \\ \texttt{rear wheel's lean a. vel.} \end{matrix}$$

- And for instance:

$$A_{stab} = \begin{pmatrix} -62.5000 & 0 & -13.2638 & 0 \\ 0 & 0 & 1.0000 & 0 \\ 225.0401 & 0.8388 & 47.4538 & 1.5245 \\ -83.1329 & 9.5182 & -22.5493 & -0.5608 \end{pmatrix}$$

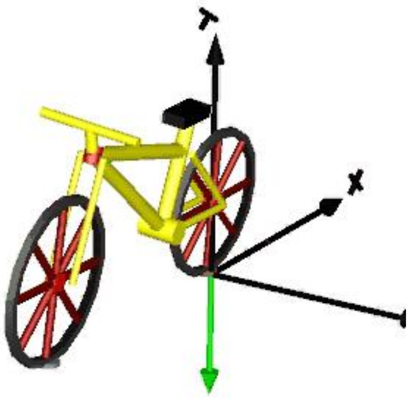
Stability Analysis

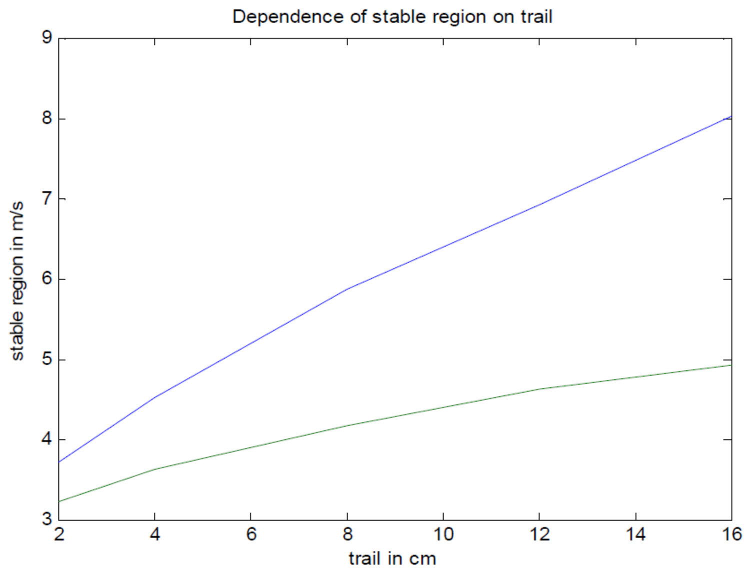
We now linearize the system for different driving velocities and perform the eigenvalue analysis



Stability Analysis

- The system is stable for driving velocities ranging from 6.1 to 10.3 m/s
- We can use this stability analysis to determine the influence of the geometry.
- For instance, let us consider the trail of the bicycle.





Questions ?